THERMAL BUCKLING ANALYSIS OF ORTHOTROPIC SHELLS

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This report contains the details of an analytical analysis of buckling of orthotropic shells. Circular cylindrical shells with clamped or simply supported boundary conditions, subjected to temperature distributions of functions of plate coordinates, are considered. By means of the strain energy method, the thermal buckling criteria of orthotropic shell may be evaluated numerically. Numerical examples are presented in this report.

This report has been prepared for the I.S.S. of the University of Cincinnati and for technically interested personnel at NASA Lewis Laboratory, Cleveland, Ohio.

<div align="right">Respectfully submitted,</div>

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ABSTRACT

The problem of thermal buckling of shells which are
made up of bi-directionally reinforced composites have
been investigated. Such shells may be treaeted as ortho-
tropic shells - a special type of anisotropic shells. The
whole investigation is devided into five sections. In
sectionI bounds on effective elastic moduli are obtained
by using the variation principle. In section II, effective
thermal constants are obtained by using the extremum
principle of thermo-elasticity. Strain energy expressions
for orthotropic shells have been derived in section III.
In deriving the strain energy expressions, non-linear
stretching of the middle surface has been taken into consi-
deration. The result has been specialised for cylindrical,
conical and spherical shells.

The section IV treats the thermal stress problems.
Two sets of equilibrium equations are derived for the case
of cylindrical and conical shells. The first set is
derived from the general equilibrium equations given by
Love. The second one is derived by setting the first variation
of potential energy equal to zero. Thermal stress problems
of cylindrical shells with various types of orthotropy
are solved for temperatures varying in axial, radial and,

in axial as well as radial direction. For a particular case, a comparison of the results obtained from two sets of equilibrium equations has been made.

Thermal buckling problems have been treated in section V. The Rayleigh-Ritz method has been used to obtain the buckling criteria. The buckling criteria contain pre-buckling rotation terms. However in numerical calculations these terms have been neglected. For cylindrical shells, the fixed end case as well as simple supported case has been considered. For conical shells only simple supported case has been investigated. Numerical results have been presented for cylindrical shells subjected to axial and radial temperature distributions.

CONTENTS

SECTION I

INTRODUCTION

I.1 Thermal Buckling of Shells

During past few decades, shell structures have been found to have extensive applications in high speed air-craft, missiles, jet engines etc. In some such applications, anisotropic material has replaced conventional isotropic material because of its more desirable properties. So the use of anisotropic shells now-a-days, is quite frequent. However, the study made on anisotropic shell structures, as compared to isotropic shells, is considerably less. One main reason may be, many more elastic constants involved, which complicate the problem to a great extent.

A seemingly important problem that may be encountered in designing an anisotropic shell structure for air-craft or missiles, is the problem of thermal buckling. Such a problem arises as the result of aerodynamic heating, caused by the super-sonic speed of these space ships. When a shell is heated in a non-uniform manner, thermal stresses develop. When there are no external forces, these stresses are self equilibrating and both tensile and compressive stresses exist. Because of the presence of compressive stresses, thermal buckling may occur.

The state of stress at which buckling occurs is referred to as critical state of stress and the corresponding temperature is called critical temperature distribution. Critical state of stress for a geometrically perfect conservative system is independant of cause of stress (1)*

Keeping this statement in mind one may see, that there are three problems that are to be solved before the buckling problem can be formulated. The first problem is of course, the determination of buckling stress. Leaving this problem aside, the other two problems are, determination of temperature distribution and the determination of thermal stresses which arise from non-uniform temperature distribution. As soon as the thermal stresses are known, the thermal buckling problem can be formulated. The temperature can be considered as characteristic parameter in the problem, since with the variation of temperature, stresses vary.

Mathematical model for thermal buckling analysis of structures is an eigen-value problem which requires the determination of a characteristic parameter that occurs in a homogeneous linear differential equation with homogeneous boundary conditions. The characteristic parameter may be determined from the condition that the non-trivial

*Number in parenthesis refers to references

solution exists. In most cases, however, the exact solution of the differential equation is not possible and so for the practical purposes approximate methods are generally used. An excellent review of some of these approximate methods, with their relative advantages and disadvantages, has been made by Pohle and Berman (2).

I.2 Object and Scope of Present Investigation

The present report is concerned with the problem of thermal stresses and thermal buckling of shells which are made up of bi-directionally reinforced composites. Such shells may be treated as orthotropic shells - a special type of anisotropic shell. Although in the present report the results are drawn for cylindrical and conical shells, the method of analysis is applicable for any shell of revolution. As mentioned earlier, one of the problems to be solved before the thermal buckling problem, is the problem of determination of temperature distribution. This is a problem of heat transfer and therefore, is not considered here. Instead of that, some arbitrary temperature distributions have been assumed and buckling criteria have been obtained for them. An effective approximate method, named Rayleigh-Ritz method has been used for solving the buckling problem. This method is based on the principle of stationary potential energy. The

potential energy of the shell is derived by assuming the shell to be made up of anisotropic homogeneous material, whose effective elastic and thermal constants are known.

The whole report consists of five parts. Parts 1 and 2 consist of the method of determination of effective elastic and thermal constants of composites. Part 3 deals with the derivation of strain energy for single and multi-layered shells considering small deflections. Part 4 deals with the thermal stress problem. The last part treats the problem of thermal buckling.

SECTION II

PREDICTION OF ELASTIC CONSTANTS OF

BI-DIRECTIONALLY REINFORCED COMPOSITES

II.1 Composite

Before discussing the methods of determination
of elastic constants of composite material, it is worth-
while to discuss something about composite in general.

Although many definitions for composites are available
in literature, they differ widely and yet there is no
commonly accepted definition. The composite material with
which we will be concerned may be defined as (3):

"A material system composed of a mixture or combination
of two or more macro constituents that differ in form and/
or material composition and that they are insoluble in one
another".

The nature of any composite depends upon the nature of
constituent materials and their shape and structural
arrangement. Where the constituent material may be metallic,
inorganic or organic and their ways of combination may be
virtually unlimited - the shape of constituents are re-
stricted to certain specific types. The major constituent
form that are used in composites are fibers, particles,
laminas, flakes, matrixes and fillers (Fig 2.1).

The matrix serves as body constituent. It encloses

the composites and gives it its bulk form. Fibers, flakes, particles etc. are structural constituents that determines the character of the internal structure of composite. Figure 2.2 shows the different types of composite slabs.

Fiber Composite

Of all the composites, fiber composite has drawn the greatest attention from structural engineers. A great improvement in strength and strength to density ratio may be obtained by combining a fibrous material of high strength and of high elastic modulus with a light weight bulk material of lower strength and lower elastic modulus. Table (1) in Reference (3) gives some idea about the degree of such improvement achievable with fiber composites. Factors that appreciably affect the mechanical and thermal properties of a fiber reinforced composite are orientation (unidirectional, bidirectional, etc), length (continuous, discontinuous), shape and composition of fibers, mechanical properties of matrix and the integrity of the bond between fiber and matrix.

II.2 Elastic Modulus of Composites - Introduction

For the analysis of composite shells, two methods are available at present: (a) Netting Analysis and (b) Ortho-tropic Analysis.

In the Netting Analysis, matrix is assumed not to
carry any load. All the loads are assumed to be carried
by the fiber, and these fibers are stressed uniformly.
This method has limited use for predicting membrane stresses
in filament wound shell and will not be considered here.

In Orthotropic Analysis, both matrix and filaments are
taken into consideration and equivalent elastic constants
are ~~found out~~ *determined*. Therefore in Orthotropic Analysis the
composite shells are treated as homogeneous anisotropic
shells having above mentioned elastic properties. In
general there are three approaches to the problem for
determining elastic properties of composites.

In the first approach, known and regular phase geo-
metries are utilized and gross approximations are made to
the nature of the stress field. It is supposed, that the
materials are made up of various combinations of simple
elements which may be in series or parallel with each other.
Many papers are available utilizing this technique. However,
such approach, is by all means, very elementary and should
not be relied upon.

In the second method, the composites are allowed to
be subjected to some simple boundary conditions. The
complete stress field is ~~found out~~ *determined* by solving the boundary
value problem. The average stress and strain in the

composite are ~~found out~~ _determined_ by considering the necessary volume

integral, and in this way the effective elastic modulus

which is the ratio of average stress and strain may be

computed. References (4-6) have used this method for

finding effective elastic constants.

In the third method, the variation principle is

utilized to compute the bounds on effective elastic moduli

in terms of strain energy. Bounds on strain energy are

obtained for simple average stress and strain field. Such

bounds on strain energy, in turn, bind effective elastic

moduli. References (7-8) have utilized this method.

II.3 Determination of Elastic Constants

II.2.1 Physical Concept: Unidirectionally reinforced

composites may, under certain condition, be treated as

homogeneous and transversly isotropic material. For

example, if we consider the reinforcement in **x**-direction

only (Fig 2.3a), we get a transverselly isotropic material

whose axis of isotropy is x-axis. Again if we consider

that the reinforcement is only in y-direction, the gross

elastic property is transversely isotropic in nature

where the axis of isotropy is in y-direction. Hence, at

this point, we can imagine bi-directionally reinforced

composite, as a composite of two phases - each phase

being transversely isotropic. The elastic constants

for the two phases may be determined by any one of the
available methods for determining elastic constants of
unidirectionally reinforced composites. It may be noted
here, in calculating the gross elastic properties of two
phases in terms of properties of fiber and matrix, only
1/2 the volume of matrix should be considered to be
associated with fibers in one direction.

II.3.2 Theoretical Analysis:

The stress strain relationships for the first and
second phase are given by (9),

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{12}\sigma_z + K_1\theta$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + K_2\theta$$

$$\varepsilon_z = a_{12}\sigma_x + a_{23}\sigma_y + a_{22}\sigma_z + K_2\theta$$

$$\gamma_{yz} = a_{44}\tau_{yz} \qquad \gamma_{xy} = a_{66}\tau_{xy} \qquad \gamma_{xz} = a_{66}\tau_{xz} \qquad (1)$$

and,

$$\varepsilon_x = a_{11}'\sigma_x + a_{12}'\sigma_y + a_{13}'\sigma_z + K_1'\theta$$

$$\varepsilon_y = a_{12}'\sigma_x + a_{22}'\sigma_y + a_{12}'\sigma_z + K_2'\theta$$

$$\varepsilon_z = a_{13}'\sigma_x + a_{12}'\sigma_y + a_{11}'\sigma_z + K_1'\theta$$

$$\gamma_{xz} = a_{55}'\tau_{xz} \qquad \gamma_{xy} = a_{66}'\tau_{xy} \qquad \gamma_{yz} = a_{66}'\tau_{yz} \qquad (2)$$

The elastic constants a_{ij} and a'_{ij} can be evaluated when the elastic properties of fiber and matrix are known. Let,

E = Young's modulus of the matrix

ν = Poisson's ratio of the matrix

E' = Young's modulus of the fiber in the X-direction.

ν' = Poisson's ratio of the fiber in the X-direction.

E" = Young's modulus of the fiber in the Y-direction

ν'' = Poisson's ratio of the fiber in the Y-direction

λ = Fiber volume in percent.

Then according to Whitney and Riley (6)

$$a_{11} = 1/\bar{E}_x$$

$$a_{12} = -\bar{\nu}/\bar{E}_x$$

$$a_{22} = 1/\bar{E}$$

$$a_{23} = -\bar{\nu}_{yz}/\bar{E}$$

$$a_{66} = 1/\bar{G}$$

$$a_{44} = 2(1-\bar{\nu}_{yz})/\bar{E} \qquad (3)$$

Where, $\bar{E}_x, \bar{\nu}, \bar{E}, \bar{\nu}_{yz}, \bar{G}$ are given in the following relations.

$$\bar{E}_x = \frac{2(\nu'-\nu)^2 E'E(1-\lambda)\lambda}{E(1-\lambda)L' + [L\lambda + (1+\nu)]E'} + E + (E'-E)\lambda \qquad (4)$$

where, $\quad L' = 1 - \nu' - 2\nu'^2 \quad ; \quad L = 1 - \nu - 2\nu^2$ $\qquad$ (5)

$$\bar{\nu} = \nu - \frac{2(\nu - \nu')(1 - \nu^2)E'\lambda}{E(1-\lambda)L' + [L\lambda + (1+\nu)E']}$$ $\qquad$ (6)

$$\bar{\nu}_{yz} = \nu'\lambda + \nu(1-\lambda)$$ $\qquad$ (7)

$$\bar{E} = [2\bar{K}(1-\bar{\nu}_{yz})\bar{E}_x] / [\bar{E}_x + 4\bar{K}\bar{\nu}^2]$$ $\qquad$ (8)

where,

$$\bar{K} = [(K'+G)K - (K'-K)G\lambda] / [(K'+G) - (K'-K)\lambda]$$ $\qquad$ (9)

K', K and G in the above expression is given by:

$$K' = E'/2(1-\nu'-2\nu'^2)$$

$$K = E/2(1-\nu-2\nu^2)$$

$$G = E/2(1+\nu)$$ $\qquad$ (10)

$$\bar{G} = [(G'+G)+(G'-G)\lambda]G / [(G'+G)-(G'-G)\lambda]$$ $\qquad$ (11)

where

$$G' = E'/2(1+\nu')$$ $\qquad$ (12)

For the elastic constants in the other phase, we have,

$$a'_{22} = 1/\bar{E}_{x_2} \qquad\qquad a_{11} = 1/\bar{E}_2$$

$$a'_{12} = -\bar{\nu}_2/\bar{E}_{x_2} \qquad\qquad a'_{13} = -\bar{\nu}_{yz_2}/\bar{E}_2$$

$$a'_{66} = 1/\bar{G}_2 \qquad\qquad a'_{55} = 2(1-\bar{\nu}_{yz2})/\bar{E}_2 \qquad\qquad (13)$$

Where, $\bar{E}_{x2}, \bar{\nu}_2, \nu_{yz2}, \bar{G}_2$ are to be found in the following way.

$$\bar{E}_{x_2} = \frac{2(\nu''-\nu)^2 E'' E (1-\lambda) \lambda}{E(1-\lambda) L'' + [L\lambda + (1+\nu)] E''} + E + (E''-E)\lambda \qquad (14)$$

Where,

$$L'' = 1 - \nu'' - 2\nu''^2 \qquad\qquad (15)$$

$$\bar{\nu}_2 = \nu - \frac{2(\nu-\nu'')(1-\nu^2) E''\lambda}{E(1-\lambda) L'' + [L\lambda + (1+\nu)]E''} \qquad\qquad (16)$$

$$\bar{\nu}_{yz2} = \nu''\lambda + \nu(1-\lambda) \qquad\qquad (17)$$

$$\bar{K}_2 = [(K''+G)K - (K''-K)G\lambda] / [(G''+G) - (G''-G)\lambda] \qquad (18)$$

Where,

$$\bar{E}_2 = [2\bar{K}_2(1-\bar{\nu}_{yz2})\bar{E}_{x2}] / [\bar{E}_{x_2} + 4\bar{K}_2\bar{\nu}_2^2] \qquad\qquad (19)$$

$$K'' = E''/2(1-\nu'' - 2\nu''^2) \qquad\qquad (20)$$

$$\bar{G}_2 = [(G''+G) + (G''-G)\lambda]G / [(G''+G) - (G''-G)\lambda] \qquad (21)$$

Let us now assume that the stress – strain relations in terms of gross composite properties are given by,

$$\mathcal{E}_x = A_{11} \sigma_x + A_{12} \sigma_y + A_{13} \sigma_z$$

$$\mathcal{E}_y = A_{12} \sigma_x + A_{22} \sigma_y + A_{23} \sigma_z$$

$$\mathcal{E}_z = A_{13} \sigma_x + A_{23} \sigma_y + A_{33} \sigma_z$$

$$\gamma_{yz} = A_{44} \tau_{yz} \qquad \gamma_{xz} = A_{55} \tau_{xz} \qquad \gamma_{xy} = A_{66} \tau_{xy} \qquad (22)$$

Now, to evaluate the bounds on A_{ij}, a small cube of composite material is considered. The strain energy of the specimen is given by:

$$U = \frac{1}{2} \int_V \left(\sigma_x \mathcal{E}_x + \sigma_y \mathcal{E}_y + \sigma_z \mathcal{E}_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx} \right) dV \qquad (23)$$

1. Bounds on A11

If the above mentioned specimen is subjected to uni-axial tension in X-direction, then the strain energy in terms of gross composite property will be

$$U_{comp} = \frac{1}{2} A_{11} \sigma^2 V \qquad (24)$$

or, $$U_{comp} = \frac{1}{2} \mathcal{E}^2 V / A_{11} \qquad (25)$$

Where, σ is the macroscopic stress on the specimen in the X-direction and A_{11} is the compliance of the composite in the X-direction. The lower and upper bound on A_{11} may be found out by using the theorem of least work and that of minimum potential energy (7).

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The strain energy of the specimen subjected to the
loading as mentioned above may be approximated by

$$U = \frac{1}{2} \int_V \sigma^2 a_{11}^\alpha \, dV \tag{26}$$

where, a_{11}^α is the summation of the product of compliance
in the X-direction and volume fraction of each phase. If
it is assumed that the fiber sizes in the X and Y - direction
are approximately the same, then,

$$a_{11}^\alpha = (a_{11} + a_{11}')/2 \tag{27}$$

Therefore, from equation (26) we get,

$$U = \frac{1}{4} \sigma^2 (a_{11} + a_{11}') \tag{28}$$

Now, from the theorem of least work actual strain energy
U_{comp} in the specimen cannot exceed U and hence, comparing
equation (24) and equation (28) we get,

$$A_{11} \leq (a_{11} + a_{11}')/2 \tag{29}$$

The lower bound of A_{11} may be obtained in the following
way. Let us suppose the cube is given a strain in the X-
direction. The strain energy for the loading may be appro-
ximated by,

$$U = \frac{1}{4} \int_V \varepsilon^2 \left(1/a_{11} + 1/a'_{11} \right) dV \qquad (30)$$

Now, from the principle of minimum potential energy,

$$U_{comp} \leqslant U$$

Comparing equation (25) and equation (30) we get,

$$A_{11} \geqslant \frac{2 a_{11} a'_{11}}{(a_{11} + a'_{11})} \qquad (31)$$

For the bounds of A_{22} ~~and A_{33}~~ the specimen is assumed to be subjected to uni-axial tension in Y ~~and Z~~-direction and the remaining procedure is the same as before. For the bounds of ~~A_{44}, A_{55} and~~ A_{66}, a shear stress or, shear strain is given ~~alternately~~ in ~~X-Z, Z-Y and~~ X-Y direction, ~~de-~~ ~~pending upon the constants whose bounds are to be evaluated.~~ Rest of the procedure is same as the first case. The following results are therefore quite obvious.

$$\frac{2 a_{22} a'_{22}}{(a_{22} + a'_{22})} \leqslant A_{22} \leqslant \frac{(a_{22} + a'_{22})}{2};$$

$$\frac{2 a_{11} a'_{11}}{(a_{11} + a'_{11})} \leqslant A_{11} \leqslant \frac{(a_{11} + a'_{11})}{2}$$

$$\frac{2 a_{66} a'_{66}}{(a_{66} + a'_{66})} \leqslant A_{66} \leqslant \frac{a_{66} + a'_{66}}{2};$$

SECTION III

THERMAL EXPANSION CO-EFFICIENT OF BI-DIRECTIONALLY REINFORCED COMPOSITES.

III.1 Introduction

Although considerable work has been done for predicting effective elastic moduli of composites in terms of its constituent material properties and volume fraction, corresponding work done for predicting the effective thermal co-efficient of the composite are comparatively small. Some work has been done in this particular field by Levin (10), Van Fo-Fy(11) of Russia and Schapery (12) of U.S.A. and few others.

Levin derived the relation between composite thermal co-efficient, composite effective moduli and constituent property. Van Fo-Fy made a detailed stress analysis for deriving effective thermal co-efficient of doubly periodic arrayed fiber reinforced composites. Schapery found the bounds on effective thermal expansion co-efficients of anisotropic composite material whose phases are isotropic by using extremum principle of thermo elasticity. Such method is also applicable for anisotropic composites with anisotropic phases. Here the method of Schapery will be extended for applying it to bi-directionally reinforced composites.

III.2 Theoretical Analysis

As it has been done previously, it will be assumed
that bi-directionally reinforced composite is a composite
made up of two transversally isotropic phases. Let us
consider a cube of specimen under isothermal condition.
We assume the cube to be approximately homogeneous. We
further assume that the phases of the composite are un-
stressed at certain temperature, when there is no external
force present. The difference from the above mentioned
temperature level will be donated by θ . At first, formulas
for effective thermal co-efficient of composites consisting
of generally anisotropic phases will be developed. This
result can be specialized for composites consisting of two
transversally isotropic phases.

The potential energy of the composite specimen is
given by,

$$V = U + \Omega$$

Where, U is the strain energy and Ω is the potential
energy due to external load.
Therefore:

$$V = \frac{1}{2} \int_V \left[\sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij} \, \varepsilon_i \varepsilon_j - \sum c_i \theta \varepsilon_i \right] dV - \int_S T_i \, u_i \, ds \qquad (1)$$

We know,

$$\sigma_i = \sum_{j=1}^{6} b_{ij} \varepsilon_i - C_i \theta \qquad (2)$$

$$\varepsilon_i = \sum_{j=1}^{6} a_{ij} \sigma_j + K_i \theta \qquad (3)$$

$$K_i = \sum_{j=1}^{6} a_{ij} C_j \qquad (4)$$

$$C_i = \sum_{j=1}^{6} b_{ij} K_j \qquad (5)$$

Applying divergence theorem we get,

$$V = \int_V \left\{ \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij} \varepsilon_i \varepsilon_j - \sum_{i=1}^{6} C_i \theta \varepsilon_i - \sum_{i=1}^{6} \varepsilon_i \sigma_i \right\} dv \qquad (6)$$

From (6) we get, the potential energy density $= -\sum_{i=1}^{6} \sigma_i K_i \theta$

$$-\frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} a_{ij} \sigma_i \sigma_j - \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij} K_i K_j \theta^2 \qquad (7)$$

Now the negative of total complimentary energy from Langhaar (14)

$$V_c = -\int_V \left[\frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} a_{ij} \sigma_i \sigma_j + \sum_{i=1}^{6} K_i \sigma_i \theta \right] dv \qquad (8)$$

$$\therefore V_c = V + \int_V \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij} K_i K_j \theta^2 dv \qquad (9)$$

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If we assume now, that the composite is elastically homogenous and anisotropic, then the stress strain relation will be given by,

$$\varepsilon_i = \sum_{j=1}^{6} A_{ij}\,\sigma_j + K_i\,\theta \qquad (10)$$

Therefore the negative of complimentary strain energy in terms of gross composite property is,

$$V_c = \int_V \left[-\frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} A_{ij}\,\sigma_i\sigma_j - \sum_{i=1}^{6} K_i\,\sigma_i\,\theta + C \right] dv \qquad (11)$$

Where, $C = C(\theta)$ is a function of temperature.

The potential energy is always minimum for all continuous displacements which satisfy boundary conditions and the negative of complimentary energy is maximum when exact stresses are used. Keeping the extremum properties in mind, we may write,

$$V_c^{\alpha} \leqslant -\frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} A_{ij}\,\sigma_i\sigma_j - \sum_{i=1}^{6} K_i\,\sigma_i\,\theta + C \leqslant V^{\alpha}$$

$$+ \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij}\,K_i K_j \qquad (12)$$

where, V^a and V_c^a are approximate values of potential and negative of complimentary energies. The bounds on thermal co-efficients will be derived, considering the inequality (12).

We will now proceed to derive the approximate expressions for energies. We assume a state of constant strain for the derivation of Potential Energy. Such strain may be obtained by assuming a surface displacement of the type,

$$U_i = e_{ij} x_j$$

Where e_{ij} is constant and $e_{ij} = e_{ji}$.
So on substitution of equation (13) in (6) we get,

$$V^a = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij}^a e_i e_j \;-\; \sum_{i=1}^{6} C_i^a \theta e_i \;-\; \sum_{i=1}^{6} e_i \sigma_i$$

$$= \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij}^a e_i e_j \;-\; \sum_{i=1}^{6} \sum_{j=1}^{6} (b_{ij} K_j)^a \theta e_i \;-\; \sum_{i=1}^{6} e_i \sigma_i$$

$$(14)$$

Where,

$$b_{ij}^a = (b_{ij})_1 \, v_1 + (b_{ij})_2 \, v_2$$

$$(b_{ij} K_j)^a = (b_{ij} K_j)_1 \, v_1 + (b_{ij} K_j)_2 \, v_2$$

$$e_{ii} = e_i \qquad \text{for } i = 1,2,3$$

$$e_{12} = e_6 \qquad\qquad e_{13} = e_5 \qquad\qquad e_{32} = e_4$$

Subscript 1 and 2 indicate the phases, v_1 and v_2 denote their volume fractions. The value of V^a will be minimum for a strain that may be determined by the equation $\frac{\partial V^a}{\partial e_i} = 0$. From the above equation we obtain,

$$\sum_{j=1}^{6} b_{ij}^{a} e_j - \sum_{j=1}^{6} (b_{ij} K_j)^a \theta - \sigma_i = 0 \tag{15}$$

Therefore,

$$e_i = B_{ij} (b_{ji} K_i)^a \theta + B_{ij} \sigma_j \tag{16}$$

Where,

$$B_{ij} = [b_{ji}^a]^{-1}$$

Substituting the strain in equation (14) we get,

$$V^a = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} b_{ij}^{a} \left[B_{il} (b_{li} K_i)^a B_{jm} \cdot \right.$$

$$(b_{mj} K_j)^a \theta^2 + B_{il} (b_{li} K_i)^a B_{jm} \sigma_m \theta + B_{jm} \theta \cdot$$

$$\left. (b_{mj} K_j)^a B_{il} \sigma_l + B_{il} B_{jm} \sigma_m \sigma_l \right] - \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6}$$

$$\left[\left(b_{ij}K_j\right)^a \theta^2 B_{il}\left(b_{li}K_i\right)^a + \left(b_{ij}K_j\right)^a \theta \, B_{il} \, \sigma_l \right]$$

$$- \sum_{i=1}^{6} \sum_{l=1}^{6} \left[B_{il}\left(b_{li}K_i\right)^a \theta \, \sigma_i - B_{il} \, \sigma_l \, \sigma_i \right] \qquad (17)$$

Now let us assume a constant stress distribution; the approximate value of V_c^a is given by,

$$V_c^a = -\frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} \bar{a}_{ij} \, \sigma_i \sigma_j - \sum_{j=1}^{6} \bar{K}_j \, \sigma_j \, \theta \qquad (18)$$

Where,

$$\bar{a}_{ij} = \int_{V=1} a_{ij} \, dv = \left(a_{ij}\right)_1 v_1 + \left(a_{ij}\right)_2 v_2$$

and

$$\bar{K}_j = \int_{V=1} K_j \, dv = \left(K_j\right)_1 v_1 + \left(K_j\right)_2 v_2$$

Where subscripts 1 and 2 denote the different phases and v_1 and v_2 denote their volume fractions.

Linear Co-Efficient of Expansion

For finding bounds on thermal co-efficient of expansion we set $\sigma_1 = \sigma$ and all other $\sigma_i = 0$.

Substituting these values in equation (17) and (18) we get,

$$V = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} b_{ij}{}^{a} \left[B_{il} (b_{li} K_i)^{a} B_{jm} (b_{mj} K_j)^{a} \theta^2 \right.$$

$$+ B_{il} (b_{li} K_i)^{a} B_{j1} \sigma_1 \theta + B_{jm} (b_{mj} K_j)^{a} \theta B_{i1} \sigma_1$$

$$\left. + B_{i1} B_{j1} \sigma^2 \right] - \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6} (b_{ij} K_j)^{a} \theta^2 B_{il} (b_{li} K_i)^{a}$$

$$- \sum_{i=1}^{6} \sum_{j=1}^{6} (b_{ij} K_j)^{a} \theta B_{i1} \sigma_1 - \sum_{l=1}^{6} B_{1l} (b_{l1} K_1)^{a} \theta \sigma_1 - B_{11} \sigma_1{}^2$$

or, $\quad V^{a} = -\frac{1}{2} \sigma^2 a_{11}^{a} - \bar{k}^{a} \sigma \theta - N_1^{a} \theta^2 \qquad$ (19)

where, $\quad a_{11}^{a} = 2 B_{11} - \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij}{}^{a} B_{i1} B_{j1}$

$$\bar{k}^{a} = \sum_{i=1}^{6} \sum_{j=1}^{6} (b_{ij} K_j)^{a} B_{i1} + \sum_{l=1}^{6} B_{1l} (b_{l1} K_1)^{a} - \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6} [b_{ij}^{a} \cdot$$

$$B_{il} (b_{li} K_i)^{a} B_{j1} + b_{ij}{}^{a} B_{jl} B_{i1} (b_{lj} K_j)^{a}$$

$$N_1^{a} = \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6} (b_{ij} K_j)^{a} B_{il} (b_{li} K_i)^{a} - \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6}$$

$$b_{ij}^{a} B_{il} (b_{li} K_i)^{a} B_{jm} (b_{mj} K_j)^{a}.$$

And,

$$V_c^{a} = -\frac{1}{2} \bar{a}_{11} \sigma^2 - \bar{K}_1 \sigma \theta \qquad (20)$$

Therefore, inequality (12) becomes,

$$-\frac{1}{2}\bar{a}_{11}\sigma^2 - \bar{K}_1\sigma\theta \;\leqslant\; -\frac{1}{2}A_{11}\sigma^2 - B_1\sigma\theta + C \;\leqslant\; -\frac{1}{2}a_{11}^a\sigma^2$$

$$-\bar{k}_1^a\sigma\theta - N_1^a\theta^2 + \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}b_{ij}K_iK_j\theta^2 \qquad (21)$$

The inequality is rewritten in order to isolate the unknown function $C = C(\theta)$.

$$-\frac{1}{2}(\bar{a}_{11} - A_{11})\sigma^2 + (K_1 - \bar{K}_1)\sigma\theta \;\leqslant\; C \;\leqslant\; \frac{1}{2}(A_{11} - a_{11}^a)\sigma^2$$

$$+(K_1 - \bar{k}_1^a)\sigma\theta - \theta^2\left[N_1^a - \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}b_{ij}K_iK_j\right] \qquad (22)$$

Maximizing the left hand side and minimizing the right hand side with respect to σ we get,

$$\sigma = \frac{(K_1 - \bar{K}_1)\theta}{(\bar{a}_{11} - A_{11})} \qquad (23a)$$

$$\sigma = -\frac{(K_1 - \bar{k}_1^a)}{(A_{11} - a_{11}^a)} \qquad (23b)$$

Putting these values in (22) we have,

$$\frac{(K_1-\bar{K}_1)^2}{(\bar{a}_{11}-A_{11})} \leqslant \frac{2C}{\theta^2} \leqslant -\frac{(K_1-\bar{k}_1^a)^2}{(A_{11}-\partial_{11}^a)} - \left(N_1^a - \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} b_{ij} K_i K_j\right) \qquad (24)$$

The function C may be dropped out from our consideration because of the fact that right hand side of inequality (24) is never less than left hand side, irrespective of the value of C.

Therefore, from inequality (24) we get,

$$(K_1 - K')(K_1 - K'') \leqslant 0$$

Where,

$$K' = \bar{K}_1 + (\bar{k}_1^a - \bar{K}_1)\frac{(\bar{a}_{11} - A_{11})}{(\bar{a}_{11} - \partial_{11}^a)} - \hbar \qquad (26)$$

$$K'' = \bar{K}_1 + (\bar{k}_1^a - \bar{K}_1)\frac{(\bar{a}_{11} - A_{11})}{(\bar{a}_{11} - \partial_{11}^a)} + \hbar \qquad (27)$$

$$\hbar = \left\{ \left[\frac{\bar{K}_1(A_{11}-\partial_{11}^a) + \bar{k}_1^a(\bar{a}_{11}-A_{11})}{(\bar{a}_{11}-\partial_{11}^a)^2}\right]^2 - \frac{(\bar{a}_{11}-A_{11})(A_{11}-a_{11}^a)}{(\bar{a}_{11}-\partial_{11}^a)} \right.$$

$$\left. \times \left[\frac{\bar{K}_1^2}{(\bar{a}_{11}-A_{11})} + \frac{\bar{k}_1^{a2}}{(A_{11}-\partial_{11}^a)} + N_1^a - \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} b_{ij} K_i K_j\right]\right\}^{1/2}$$

$$\qquad (28)$$

Inequality (24) is satisfied only when $K' \leqslant K_1 \leqslant K''$ and hence K', K'' are lower and upper bound of effective linear thermal co-efficient of composite in X-direction.

SECTION IV

STRAIN ENERGY EXPRESSIONS

IV.1 Introduction

It has been pointed out earlier that in the present investigation, an orthotropic analysis will be made; i.e., the composite shell will be treated as a shell made up of homogeneous anisotropic material. For the case, where the reinforcements are in two perpendicular directions, the composite exhibits the property of a special type of anisotropy - called orthotropy. Hence for finding out the strain energy expression of shells made up of bi-directionally reinforced composite, it is required to find first, the strain energy of an orthotropic shell. In the expression of strain energy so obtained, the elastic constants are to be replaced by the effective elastic constants of the composites. However the same problem can be attacked in another way. The shell mentioned above is, in actuality, a multi-layered shell where each layer is unidirectionally reinforced. Therefore the strain energy of the shell will be that of multi-layered anisotropic shell where each layer is transversely isotropic. Hence in the following pages, the strain energy expression for the single layered anisotropic as well as multi-layered anisotropic shell will be derived and the results will be

specialized for the particular cases.

IV.2 Brief Review of Geometry of Shells

IV.2.1 Surface theory: A surface may be defined by
the equation of the type $X = X(x,y)$, $Y = Y(x,y)$, $Z = Z(x,y)$,
in which X, Y, Z are rectangular coordinates and x, y are
parameters which are called surface coordinates(13). A
point can be located by the following vector

$$\bar{r} = Xi + Yj + Zk \tag{1}$$

where i, j, k are unit vectors along X, Y and Z direction.
A surface can be represented by the vector equation

$$\bar{r} = \bar{r}(x,y)$$

Surface coordinates x, y are orthogonal if $\bar{r}_x \cdot \bar{r}_y = 0$
where $\bar{r}_x(\bar{r}_y)$ is partial derivative of the position vector
with respect to x(y).

The square of the distance between two neighboring
points with the surface coordinate (x,y) and (x+dx,y+dy) is
given by

$$ds^2 = d\bar{r} \cdot d\bar{r} = A^2 dx^2 + B^2 dy^2 \tag{2}$$

where

$$A^2 = \bar{r}_x \cdot \bar{r}_x = X_x^2 + Y_x^2 + Z_x^2$$

$$B^2 = \bar{r}_y \cdot \bar{r}_y = X_y^2 + Y_y^2 + Z_y^2 \tag{3}$$

From the equation (3) it is seen that the magnitude of r_x and r_y are A and B respectively. So the area of the surface is given by $\iint AB \, dx dy$.

From the definition of cross-product the unit vector, normal to the surface is found to be

$$\hat{n} = \frac{\bar{r}_x \times \bar{r}_y}{AB} \tag{4}$$

Second Fundamental Form of Surfaces

$$e dx^2 + 2f dx dy + g dy^2 = -d\bar{r} \cdot d\hat{n} \tag{5}$$

For orthogonal surface coordinate, e, f, g are given by

$$e = \bar{r}_{xx} \cdot \hat{n} = \bar{r}_{xx} \cdot (\bar{r}_x \times \bar{r}_y)/AB$$

$$f = \bar{r}_{xy} \cdot \hat{n} = \bar{r}_{xy} \cdot (\bar{r}_x \times \bar{r}_y)/AB$$

$$g = \bar{r}_{yy} \cdot \hat{n} = \bar{r}_{yy} \cdot (\bar{r}_x \times \bar{r}_y)/AB \tag{6}$$

Extreme values of curvature are denoted by $1/r_1$ and $1/r_2$ and are known as principal curvatures. Lines of principal curvature coincide with coordinate lines, only when the coordinates are orthogonal and f=o. In that case

$$\frac{1}{r_1} = -e/A^2$$

$$\frac{1}{r_2} = -g/B^2 \tag{7}$$

Theorem of Rodrigues

When the line of principal curvature coincide with coordinate lines, then

$$\hat{n}_x = \overline{r}x/r_1$$

$$\hat{n}_y = \overline{r}y/r_2 \tag{8}$$

IV.3 Geometric Representation of Shells

Let the middle surface of the shell be represented by X=X(x,y), Y=Y(x,y), Z=Z(x,y), where x,y are orthogonal surface co-ordinates. Let ±z be measured from this surface; positive z is measured in the positive sense of the surface normal (see equation 4). Let the free surface of an un-deformed shell be represented by the surfaces z = ±h/2,

where h is the thickness of the shell which may be a function of x,y or, constant.

The surface z = constant in the undeformed shell is defined by the equation

$$\bar{R} = \bar{r} + z\hat{n} \tag{9}$$

Differentiating with respect to x and y one obtains,

$$\bar{R}_x = \bar{r}_x + \hat{n}_x z \qquad\qquad \bar{R}_y = \bar{r}_y + \hat{n}_y z \tag{10}$$

respectively,

From the theorem of Dupin,$^{(14)}$ it may be seen that if the shell co-ordinates are orthogonal, the co-ordinate lines on the middle surface must be lines of principle curvature. Hence Rodrigues theorem can be applied. From equation (10) and (8) one obtains,

$$\bar{R}_x = \left(1 + z/r_1\right)\bar{r}_x \qquad \bar{R}_y = \left(1 + z/r_2\right)\bar{r}_y \qquad \bar{R}_z = \hat{n} \tag{11}$$

Now,

$$ds^2 = d\bar{R} \cdot d\bar{R} = \left(\bar{R}_x dx + \bar{R}_y dy + \bar{R}_z dz\right)^2 \tag{12}$$

From equation (3), (11) and from the fact, $\bar{r}_x \cdot \hat{n} = \bar{r}_y \cdot \hat{n} = \bar{r}_x \cdot \bar{r}_y = 0$,

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2 \tag{13}$$

where,

$$\alpha = A\left(1 + z/r_1\right) \qquad \beta = B\left(1 + z/r_2\right) \qquad \gamma = 1$$

(14)

α, β and γ are called Lame' co-efficients.

IV.4 Strain Energy Expression for Orthotropic Material

For convenience let strains in the directions x,y, etc. be represented by $\varepsilon_1, \varepsilon_2$ etc. respectively, i.e $\varepsilon_x = \varepsilon_1$, $\varepsilon_y = \varepsilon_2$, $\varepsilon_z = \varepsilon_3$, $\gamma_{yz} = \varepsilon_4$, $\gamma_{zx} = \varepsilon_5$, $\gamma_{xy} = \varepsilon_6$.

The strain energy density of a hookean material is given by (2)

$$U_o = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij}\, \varepsilon_i \varepsilon_j - \sum_{i=1}^{6}(C_i\theta + g_i)\varepsilon_i + Const.$$

(15)

where b_{ij}'s are called moduli of elasticity (9), C_i's are thermal constants and θ is the temperature. If the origin of strain is so chosen that $g_i = 0$ and neglecting the constant term, we get,

$$U_o = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} b_{ij}\, \varepsilon_i \varepsilon_j - \sum_{i=1}^{6} C_i\theta\, \varepsilon_i$$

(16)

For orthotropic material (9)

$$b_{14} = b_{24} = b_{34} = b_{46} = b_{15} = b_{25} = b_{35} = b_{56} = b_{16} =$$

$$b_{26} = b_{36} = b_{45} = C_4 = C_5 = C_6 = 0. \qquad (17)$$

Hence the expression for strain energy density

$$U_0 = \frac{1}{2} \left[b_{11} \mathcal{E}_1^2 + 2b_{12} \mathcal{E}_1 \mathcal{E}_2 + 2b_{13} \mathcal{E}_1 \mathcal{E}_3 + b_{22} \mathcal{E}_2^2 + b_{66} \mathcal{E}_6^2 + b_{33} \mathcal{E}_3^2 \right.$$

$$\left. + 2b_{23} \mathcal{E}_2 \mathcal{E}_3 + b_{44} \mathcal{E}_4^2 + b_{55} \mathcal{E}_5^2 \right] - \theta \left[C_1 \mathcal{E}_1 + C_2 \mathcal{E}_2 + C_3 \mathcal{E}_3 \right]$$

$$(18)$$

IV.5 Strain Energy of an Orthotropic Shell

For a shell co-ordinate the strain displacement relations are approximately given by (14),

$$\mathcal{E}_x = \frac{1}{\alpha} \left(u_x + \alpha_y v/\beta + \alpha_z \omega \right) + \frac{\omega_x^2}{2A^2}$$

$$\mathcal{E}_y = \frac{1}{\beta} \left(\beta_x u/\alpha + v_y + \beta_z \omega \right) + \frac{\omega_y^2}{2B^2}$$

$$\gamma_{yz} = \frac{\omega_y}{\beta} + v_z - \frac{\beta_z v}{\beta}$$

$$\gamma_{zx} = u_z + \frac{\omega_x}{\alpha} - \frac{\alpha_z u}{\alpha}$$

$$\gamma_{xy} = \frac{u_y}{\beta} + \frac{v_x}{\alpha} - \frac{\beta_x v}{\alpha \beta} - \frac{\alpha_y u}{\alpha \beta} + \frac{\omega_x \omega_y}{AB} \qquad (19)$$

Where u,v and w are displacements in x,y and z directions.

Now it is assumed that the transverse shearing stresses τ_{xz}, τ_{yz} vanish which gives $\gamma_{xz} = \gamma_{yz} = 0$. Also it is assumed that ω does not vary much with z and hence is a function of x and y only. So from equation (19) we get,

$$\frac{\partial}{\partial z}\left(\frac{u}{\alpha}\right) + \frac{\omega_x}{\alpha^2} = 0 \qquad\qquad \frac{\partial}{\partial z}\left(\frac{v}{\beta}\right) + \frac{\omega_y}{\beta^2} = 0 \qquad\qquad (20)$$

Noting from equation (14) $\quad \alpha = A\left(1 + z/r_1\right)$, $\beta = B\left(1 + z/r_2\right)$ and $\gamma = 1$, we get from eqn. (20),

$$u = \frac{\omega_x r_1}{A} + \alpha f(x,y) \qquad\qquad v = \frac{\omega_y r_2}{B} + \beta g(x,y) \qquad\qquad (21)$$

From the condition $u = \bar{u}$ and $v = \bar{v}$ at $z = 0$ we get,

$$u = \frac{1}{A}\left(\alpha\bar{u} - z\omega_x\right) \qquad\qquad v = \frac{1}{B}\left(\beta\bar{v} - z\omega_y\right) \qquad\qquad \omega = \bar{\omega}$$

$$(23)$$

From equation (19) and (23) we get the strain components as

$$\varepsilon_x = \frac{1}{\alpha}\frac{\partial}{\partial x}\left[\frac{\alpha\bar{u} - z\omega_x}{A}\right] + \frac{\alpha_y}{\alpha}\left[\frac{\beta\bar{v} - z\omega_y}{\beta B}\right] + \frac{\alpha_z \omega}{\alpha} + \frac{\omega_x^2}{2A^2}$$

$$\varepsilon_y = \frac{1}{\beta}\frac{\partial}{\partial y}\left[\frac{\beta\bar{v}-z\omega_y}{B}\right] + \frac{\beta_x}{\beta}\left[\frac{\alpha\bar{u}-z\omega_x}{\alpha A}\right] + \frac{\beta_z\omega}{\beta} + \frac{\omega_y^2}{2B^2}$$

$$\gamma_{xy} = \frac{\beta}{\alpha}\frac{\partial}{\partial x}\left[\frac{\beta\bar{v}-z\omega_y}{\beta B}\right] + \frac{\alpha}{\beta}\frac{\partial}{\partial y}\left[\frac{\alpha\bar{u}-z\omega_x}{\alpha A}\right] + \frac{\omega_x\omega_y}{AB}$$

(24)

Love (15) proposed that equations (24) can be linearized in z. Then,

$$\varepsilon_x = e_x + z K_x \qquad \varepsilon_y = e_y + z K_y \qquad \gamma_{xy} = e_{xy} + z K_{xy} \quad (25)$$

Where e_x, e_y and e_{xy} are the values of ε_x, ε_y and ε_{xy} on the middle surface of the shell.

Setting z=0, we see from equation (24)

$$e_x = \frac{\bar{u}_x}{A} + \frac{\bar{v}A_y}{AB} + \frac{\omega}{r_1} + \frac{\omega_x^2}{2A^2}$$

$$e_y = \frac{\bar{v}_y}{B} + \frac{\bar{u}B_x}{AB} + \frac{\omega}{r_2} + \frac{\omega_y^2}{2B^2}$$

$$e_{xy} = \frac{\bar{v}_x}{A} + \frac{\bar{u}_y}{B} - \frac{A_y\bar{u}}{AB} - \frac{B_x\bar{v}}{AB} + \frac{\omega_x\omega_y}{AB}$$

(26)

K_x, K_y, K_{xy} are obtained from equation (24) and (25) and they may be approximated (14) as

$$K_x = -\frac{1}{A}\frac{\partial}{\partial x}\left(\frac{\omega_x}{A}\right) - \frac{A_y \omega_y}{AB^2}$$

$$K_y = -\frac{B_x \omega_x}{A^2 B} - \frac{1}{B}\frac{\partial}{\partial y}\left(\frac{\omega_y}{B}\right)$$

$$K_{xy} = \frac{2}{AB}\left(\frac{A_y \omega_x}{A} + \frac{B_x \omega_y}{B} - \omega_{xy}\right) \tag{27}$$

Now putting the values of e_x, e_y etc. in the equation (18) the strain energy density becomes:

$$U_0 = \frac{1}{2}\left[b_{11}(e_x + zK_x)^2 + 2b_{12}(e_x + zK_x)(e_y + zK_y) + b_{22}.\right.$$

$$\left.(e_y + zK_y)^2 + b_{66}(e_{xy} + zK_{xy})^2\right] - \theta\left[C_1(e_x + zK_x) + C_2(e_y + zK_y)\right] \tag{28}$$

Now, if the volume element of the shell $\alpha\beta\, dx\, dy\, dz$ is approximated by $AB\, dx\, dy\, dz$ then,

Total potential energy $\displaystyle U = \iint AB\, dx\, dy \int_{-h/2}^{h/2} U_0\, dz$

$$= \frac{1}{2}\iint\left(b_{11}e_x^2 + b_{22}e_y^2 + b_{66}e_{xy}^2 + 2b_{12}e_x e_y\right) hAB\, dx\, dy$$

$$+ \frac{1}{24}\iint\left(b_{11}K_x^2 + b_{66}K_{xy}^2 + b_{22}K_y^2 + 2b_{12}K_x K_y\right) h^3 AB\, dx\, dy$$

$$- \iint\left[(C_1 e_x + C_2 e_y)\theta_0 + (C_1 K_x + C_2 K_y)\theta_1\right] AB\, dx\, dy \tag{29}$$

where

$$\theta_o = \int_{-h/2}^{h/2} \theta \, dz$$

$$\theta_1 = \int_{-h/2}^{h/2} \theta z \, dz \qquad\qquad (30)$$

To check that this expression reduces to standard expression of isotropic case we proceed as follows.

To evaluate b_{11}, b_{22}, b_{12} etc. in terms of a_{ij}'s , we observe the following relations:

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + K_1\theta$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + K_2\theta$$

From which we obtain,

$$\sigma_x = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}\,\varepsilon_x + \frac{a_{12}}{a_{12}^2 - a_{11}a_{22}}\varepsilon_y - \theta\left(\frac{K_1 a_{22} - K_2 a_{12}}{a_{11}a_{22} - a_{12}^2}\right)$$

$$\sigma_y = \frac{a_{12}}{a_{12}^2 - a_{11}a_{22}}\,\varepsilon_x + \frac{a_{11}}{a_{11}a_{22} - a_{12}^2}\,\varepsilon_y - \theta\left(\frac{K_1 a_{12} - K_2 a_{11}}{a_{12}^2 - a_{11}a_{22}}\right)$$

Hence,

$$b_{11} = a_{22} / (a_{11}a_{22} - a_{12}^2)$$

$$b_{12} = a_{12} / (a_{12}^2 - a_{11}a_{22})$$

$$b_{22} = a_{11}/(a_{11}a_{22} - a_{12}^2)$$

$$C_1 = (K_1 a_{22} - K_2 a_{12})/(a_{11}a_{22} - a_{12}^2)$$

$$C_2 = (K_1 a_{12} - K_2 a_{11})/(a_{12}^2 - a_{11}a_{22})$$

For isotropic case,

$$a_{11} = 1/E \qquad a_{12} = -\nu/E \qquad a_{22} = 1/E \qquad K_1 = K_2 = K$$

Therefore,

$$b_{11} = b_{22} = E/(1-\nu^2)$$

$$b_{12} = \nu E/(1-\nu^2)$$

$$b_{66} = G$$

$$C_1 = C_2 = KE/(1-\nu)$$

Substituting these values of b_{11}, b_{22} etc. in the equation (29) we obtain the strain energy expression,

$$U = \frac{G}{(1-\nu)} \iint \left(e_x^2 + e_y^2 + \frac{1}{2}(1-\nu)e_{xy}^2 + 2\nu e_x e_y \right) h\, AB\, dx\, dy$$

$$+ \frac{G}{(1-\nu)} \iint \left(K_x^2 + K_y^2 + 2\nu K_x K_y + \frac{1}{2}(1-\nu) K_{xy}^2 \right) h^3 AB\, dx\, dy$$

$$- \frac{E}{(1-\nu)} \iint \left[(e_x + e_y)\theta_o + (K_x + K_y)\theta_1 \right] AB\, dx\, dy .$$

The above expression for strain energy is identical to that given by Langhaar (14) for isotropic case.

IV.5.1 Strain Energy Expression of An Orthotropic Cylindrical Shell

Let X,Y, Z be rectangular co-ordinate. The middle surface of a cylincrical shell is defined by

$$X = x \qquad Y = a\sin y \qquad Z = a\cos y$$

where, x and y are surface co-ordinates as shown in Figure (4.2).

From equation (3), (6) and (7)

$$A = 1 \qquad B = a \qquad r_1 = \infty \qquad r_2 = a$$

Strains of the middle surface of a cylindrical shell as obtained from equation (26) are,

$$e_x = \bar{u}_x + \frac{\omega_x^2}{2}$$

$$e_y = \frac{\bar{v}_y + \omega}{a} + \frac{\omega_y^2}{2a^2}$$

$$e_{xy} = \bar{v}_x + \frac{\bar{u}_y}{a} + \frac{\omega_x \omega_y}{a}$$

(31)

And from equation (27), for a cylindrical shell,

$$K_x = - \omega_{xx}$$

$$K_y = - \frac{\omega_{yy}}{a^2}$$

$$K_{xy} = - \frac{2 \omega_{xy}}{a} \tag{32}$$

Putting the values of e_x, e_y, K_x, K_y etc. from equations
(31) and (32) into the strain energy expressions given by
equation (29) we get,

Strain energy $U = \frac{1}{2} \iint \left[b_{11} \left(\bar{u}_x + \frac{\omega_x^2}{2} \right)^2 + b_{22} \left(\frac{\bar{v}_y + \omega}{a} \right. \right.$

$$\left. + \frac{\omega_y^2}{2a^2} \right)^2 + b_{66} \left(\bar{v}_x + \frac{\bar{u}_y}{a} + \frac{\omega_x \omega_y}{a} \right)^2 + 2 b_{12} \left(\bar{u}_x + \frac{\omega_x^2}{2} \right) \times$$

$$\left. \left(\frac{\bar{v}_y + \omega}{a} + \frac{\omega_y^2}{2a^2} \right) \right] h \, a \, dx \, dy$$

$$+ \frac{1}{24} \iint \left[b_{11} \omega_{xx}^2 + \frac{4 b_{66}}{a^2} \omega_{xy}^2 \right.$$

$$+ b_{22} \frac{\omega_{yy}^2}{a^4} + 2 b_{12} \frac{\omega_{xx} \omega_{yy}}{a^2} \right] h^3 a \, dx \, dy$$

$$-\iint \left\{ \left[C_1 \left(\bar{u}_x + \frac{\omega_x^2}{2} \right) + C_2 \left(\frac{\bar{v}_y + \omega}{a} + \frac{\omega_y^2}{2a^2} \right) \right] \theta_0 \right.$$

$$\left. - \left[C_1 \omega_{xx} + C_2 \frac{\omega_{yy}}{a^2} \right] \theta_1 \right\} a\,dx\,dy. \qquad (33)$$

IV.5.2 Strain Energy Expression of an Orthotropic Conical Shell

Let X,Y, Z be rectangular co-ordinates. The middle surface of a conical shell with the vertex at the origin is given by the equations (see Figure 4.3)

$$X = x\cos\alpha \qquad Y = x\sin\alpha\sin y \qquad Z = x\sin\alpha\cos y \qquad (34)$$

From equation (3), (6) and (7), we get,

$$A = 1 \qquad B = x\sin\alpha \qquad r_1 = \infty \qquad r_2 = x\tan\alpha \qquad (35)$$

The strains of the middle surface of circular conical shell as obtained from equation (26) are

$$e_x = \bar{u}_x + \frac{\omega_x^2}{2}$$

$$e_y = \frac{v_y}{x\sin\alpha} + \frac{\bar{u}}{x} + \frac{\omega}{x\tan\alpha} + \frac{\omega_y^2}{2x^2\sin^2\alpha}$$

$$e_{xy} = \bar{v}_x + \frac{\bar{u}_y}{x \sin\alpha} - \frac{\bar{v}}{x} + \frac{\omega_x \omega_y}{x \sin\alpha} \tag{36}$$

From equation (27) we get,

$$K_x = -\omega_{xx}$$

$$K_y = -\frac{\omega_x}{x} - \frac{\omega_{yy}}{x^2 \sin^2\alpha}$$

$$K_{xy} = \frac{2\omega_y}{x^2 \sin\alpha} - \frac{2\omega_{xy}}{x \sin\alpha}$$

Putting the values of e_x, e_y, K_x, K_y etc. from equation (31) and (32) into the equation (29), we get,

Strain energy $U = \frac{1}{2} \iint \left[b_{11} \left(\bar{u}_x + \frac{\omega_x^2}{2} \right)^2 + b_{22} \left(\frac{\bar{v}_y}{x \sin\alpha} \right. \right.$

$$\left. + \frac{\bar{u}}{x} + \frac{\omega}{x \tan\alpha} + \frac{\omega_y^2}{2x^2 \sin^2\alpha} \right)^2 + b_{66} \left(\bar{v}_x + \frac{\bar{u}_y}{x \sin\alpha} - \frac{v}{x} \right.$$

$$\left. + \frac{\omega_x \omega_y}{x \sin\alpha} \right)^2 + 2 b_{12} \left(\bar{u}_x + \frac{\omega_x^2}{2} \right) \left(\frac{v_y}{x \sin\alpha} + \frac{u}{x} + \right.$$

$$\left. \left. \frac{\omega}{x \tan\alpha} + \frac{\omega_y^2}{2x^2 \sin^2\alpha} \right) \right] h \, x \sin\alpha \, dx \, dy.$$

$$+ \frac{1}{24} \iint \left[b_{11} \omega_{xx}^2 + b_{66} \left(\frac{2\omega_y}{x^2 \sin\alpha} - \frac{2\omega_{xy}}{x\sin\alpha} \right)^2 + b_{22} \left(\frac{\omega_x}{x} + \right. \right.$$

$$\left. + \frac{\omega_{yy}}{x^2 \sin^2\alpha} \right)^2 + 2 b_{12} \omega_{xx} \left(\frac{\omega_x}{x} + \frac{\omega_{yy}}{x^2 \sin^2\alpha} \right) \Big] h^3 x \sin\alpha \, dx\, dy$$

$$- \iint \left\{ \left[C_1 \left(\bar{u}_x + \frac{\omega_x^2}{2} \right) + C_2 \left(\frac{v_y}{x\sin\alpha} + \frac{\bar{u}}{x} + \frac{\omega}{x\tan\alpha} + \right. \right. \right.$$

$$\left. \left. \frac{\omega_y^2}{2x^2 \sin^2\alpha} \right) \right] \theta_0 - \left[C_1 \omega_{xx} + C_2 \left(\frac{\omega_x}{x} + \frac{\omega_{yy}}{x^2 \sin^2\alpha} \right) \right] \theta_1 \right\} x \sin\alpha$$

$$. dx\, dy.$$

$$(38)$$

IV.5.3 Strain Energy Expression of an Orthotropic Spherical Shell

The middle surface of the spherical shell with the center at origin is given by the equation (See Figure 4.4),

$$X = a\sin x \sin y \qquad Y = a\sin x \cos y \qquad Z = a\cos x$$

where, a is the radius of the middle surface, x is the colatitude and y is the longitude.

For spherical shells,

$$A = a \qquad B = a\sin x \qquad r_1 = r_2 = -a$$

Therefore, the strains of the middle surface of the spherical shell are given by

$$e_x = \frac{\bar{u}_x}{a} - \frac{\omega}{a} + \frac{\omega_x^2}{2a^2}$$

$$e_y = \frac{\bar{v}_y}{a\sin x} + \frac{\bar{u}\cos x}{a\sin x} - \frac{\omega}{a} + \frac{\omega_y^2}{2a^2\sin^2 x}$$

$$e_{xy} = \frac{\bar{v}_x}{a} + \frac{\bar{u}_y}{a\sin x} - \frac{\bar{v}\cos x}{a\sin x} + \frac{\omega_x \omega_y}{a^2\sin x}$$

K_x, K_y and K_{xy} are given by

$$K_x = -\frac{\omega_{xx}}{a^2}$$

$$K_y = -\frac{\cos x}{a^2\sin x}\omega_x - \frac{\omega_{yy}}{a^2\sin^2 x}$$

$$K_{xy} = \frac{2}{a^2\sin x}\left[\frac{\cos x}{\sin x}\omega_y - \omega_{xy}\right]$$

From equation (29) strain energy for an orthotropic spherical shell are found to be $U = \frac{1}{2}\iint\left[b_{11}\left(\frac{\bar{u}_x - \omega}{a}\right.\right.$

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$$+ \frac{\omega_x^2}{2a^2}\Bigg)^2 + b_{22}\left(\frac{\bar{v}_y}{a} + \frac{\bar{u}\cot x - \omega}{a} + \frac{\omega_y^2}{2a^2\sin^2 x}\right)^2$$

$$+ b_{66}\left(\frac{\bar{v}_x}{a} + \frac{\bar{u}_y - \bar{v}\cos x + \omega_x\omega_y/a}{a\sin x}\right)^2 + 2b_{12}\left(\frac{\bar{u}_x - \omega}{a} + \frac{\omega_x^2}{2a^2}\right)$$

$$\left(\frac{\bar{v}_y}{a} + \frac{\bar{u}\cot x - \omega}{a} + \frac{\omega_y^2}{2a^2\sin^2 x}\right)\Bigg] h a^2 \sin x\, dx\, dy$$

$$+ \frac{1}{24} \iint\Bigg[b_{11}\frac{\omega_{xx}^2}{a^4} + 4b_{66}\frac{(\cot x\,\omega_y - \omega_{xy})^2}{a^4\sin^2 x} + b_{22}\left(\frac{\omega_x\cot x}{a^2}\right.$$

$$\left. + \frac{\omega_{yy}}{a^2\sin^2 x}\right)^2 + 2b_{12}\left(\frac{\omega_x\cot x}{a^2} + \frac{\omega_{yy}}{a^2\sin^2 x}\right)\frac{\omega_{xx}}{a^2}\Bigg] h^3 a^2\sin x\, dx\, dy$$

$$+ \iint\Bigg\{\left[C_1\left(\frac{\bar{u}_x - \omega}{a} + \frac{\omega_x^2}{2a^2}\right) + C_2\left(\frac{\bar{v}_y}{a\sin x} + \frac{\bar{u}\cot x - \omega}{a}\right.\right.$$

$$\left.\left. + \frac{\omega_y^2}{2a^2\sin^2 x}\right)\right]\theta_0 - \left[C_1\frac{\omega_{xx}}{a^2} + C_2\left(\frac{\omega_x\cot x}{a^2} + \frac{\omega_{yy}}{a^2\sin^2 x}\right)\right]\theta_1\Bigg\}$$

$$\times\ a^2\sin^2 x\, dx\, dy\ .$$

IV.6 Strain Energy Expression for a Transversely Isotropic Material

For transversely isotropic material we have the following relations (9)

$$b_{14} = b_{15} = b_{16} = b_{24} = b_{25} = b_{26} = b_{34} = b_{35} = b_{36}$$

$$= b_{45} = b_{46} = b_{56} = 0 \; ;$$

Also,

$$b_{66} = b_{44} \quad , \quad b_{11} = b_{33} \quad , \quad b_{23} = b_{12} \; , \quad b_{55} = 2(b_{11} - b_{13})$$

So from equation (16) strain energy density expression for transversely isotropic material becomes,

$$U_o = \frac{1}{2} \left[b_{11} (\mathcal{E}_1^2 + \mathcal{E}_3^2) + 2b_{13} \, \mathcal{E}_1 \mathcal{E}_3 + 2b_{12} (\mathcal{E}_1 \mathcal{E}_2 + \mathcal{E}_2 \mathcal{E}_3) \right.$$

$$\left. + b_{22} \mathcal{E}_2^2 + b_{44} (\mathcal{E}_4^2 + \mathcal{E}_6^2) + b_{55} \, \mathcal{E}_5^2 \right] - \theta \left[c_1 \right.$$

$$\left. (\mathcal{E}_1 + \mathcal{E}_3) + c_2 \mathcal{E}_2 \right]$$

IV.7 Strain Energy Expression of a Transversely Isotropic Cylindrical Shell

A look at the equation (18) reveals that strain energy density for transversely isotropic shells will be the same as that for orthotropic shells.

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IV.8 Strain Energy of a Multi-Layer Anisotropic Shell

The anisotropic shell considered here is assumed to be made up of N number of orthotropic layers. One of the axes of symmetry of these layers is always normal to the surface of the shell; others are arbitrarily oriented. Let us assume that the axes of symmetry of any particular layer α, β make φ with x and y respectively.

Assuming a state of generalised plane stress in the nth Lamina of the shell, the stress-strain relationship will be given by

$$
\begin{bmatrix} \sigma_1^{(n)} \\[6pt] \sigma_2^{(n)} \\[6pt] \sigma_6^{(n)} \end{bmatrix}
=
\begin{bmatrix} b_{11}^{(n)} & b_{12}^{(n)} & b_{16}^{*(n)} \\[6pt] b_{12}^{(n)} & b_{22}^{(n)} & b_{26}^{*(n)} \\[6pt] b_{16}^{*(n)} & b_{26}^{*(n)} & b_{66}^{(n)} \end{bmatrix}
\begin{bmatrix} \varepsilon_1^{(n)} - c_1^{(n)}\theta \\[6pt] \varepsilon_2^{(n)} - c_2^{(n)}\theta \\[6pt] \varepsilon_3^{(n)} - c_6^{(n)}\theta \end{bmatrix}
$$

where subscripts 1, 2 and 6 to the stress-strain tensor indicates corresponding quantities in x,y and xy directions of the shell co-ordinate respectively.

$b_{ij}^{(n)}$ can be obtained from b_{ij}'s (elastic co-efficients in the principal direction) by the transformation law given in Reference (9).

*These terms are zero for orthotropic or transversely
 isotropic layer.

Thermal co-efficients $K_1^{(n)}$, $K_2^{(n)}$ and $K_6^{(n)}$ can be obtained from the following relations.

$$K_1^{(n)} = K_1 \cos^2\varphi + K_2 \sin^2\varphi$$

$$K_2^{(n)} = K_1 \sin^2\varphi + K_2 \cos^2\varphi$$

$$K_6^{(n)} = (K_2 - K_1) \sin\varphi \cos\varphi$$

where K_1, K_2 and K_6 are thermal co-efficients in the principal directions.

Total potential energy of nth layer is given by

$$U_n = \iint AB \, dx\, dy \int_{h_{n-1}}^{h_n} U_0 \, dz$$

$$= \frac{1}{4} \iint \left[b_{11}^{(n)} e_x^2 + b_{22}^{(n)} e_y^2 + b_{66}^{(n)} e_{xy}^2 + 2 b_{12}^{(n)} e_x e_y \right.$$

$$\left. + 2 b_{16}^{(n)} e_x e_{xy} + 2 b_{26}^{(n)} e_{xy} e_y \right] \left[h_n - h_{n-1} \right] AB \, dx\, dy$$

$$+ \frac{1}{6} \iint \left[b_{11}^{(n)} K_x^2 + b_{22}^{(n)} K_y^2 + b_{66}^{(n)} K_{xy}^2 \right.$$

$$+ 2b_{12}^{(n)} K_x K_y + 2b_{16}^{(n)} K_x K_{xy} + 2b_{26} K_y K_{xy}] [h_n^3 - h_{n-1}^3]$$

$$\times A B \, dx \, dy.$$

$$- \iint \left[(C_1^{(n)} e_x + C_2^{(n)} e_y + C_6^{(n)} e_{xy}) \theta_0 + (C_1^{(n)} K_x + C_2^{(n)} K_y \right.$$

$$\left. + C_6^{(n)} K_{xy}) \theta_1 \right] A B \, dx \, dy$$

where,

$$\theta_0 = \int_{h_{n-1}}^{h_n} \theta \, dz$$

$$\theta_1 = \int_{h_{n-1}}^{h_n} \theta z \, dz$$

Total strain energy of the shell will be given by

$$U = \sum_{n=1}^{N} U_n$$

THERMAL STRESSES IN SHELL

V.1 Equilibrium Relations

The following stress notations are used. The stress σ_x is normal to a plane perpendicular to x-axis; the stresses τ_{xy} and τ_{xz} are tangent to this plane and directed in y and z directions, respectively.

Love (15) derived the differential equation for any orthogonal co-ordinates. For shell co-ordinates, in the absence of body forces, Love's equation becomes,

$$\frac{\partial}{\partial x}(\beta \sigma_x) + \frac{\partial}{\partial y}(\alpha \tau_{xy}) + \frac{\partial}{\partial z}(\alpha\beta \tau_{xz}) + \frac{\partial \alpha}{\partial y}\tau_{xy}$$

$$+ \beta \frac{\partial \alpha}{\partial z}\tau_{xz} - \frac{\partial \beta}{\partial x}\sigma_y = 0$$

$$\frac{\partial}{\partial x}(\beta \tau_{xy}) + \frac{\partial}{\partial y}(\alpha \sigma_y) + \frac{\partial}{\partial z}(\alpha\beta \tau_{yz}) + \frac{\partial \beta}{\partial z}\tau_{xy}$$

$$+ \alpha \frac{\partial \beta}{\partial z}\tau_{yz} - \frac{\partial \alpha}{\partial y}\sigma_x = 0$$

$$\frac{\partial}{\partial x}(\beta \tau_{xz}) + \frac{\partial}{\partial y}(\alpha \tau_{yz}) + \frac{\partial}{\partial z}(\alpha\beta \sigma_z) - \beta \frac{\partial \alpha}{\partial z}\sigma_x$$

$$- \alpha \frac{\partial \beta}{\partial z}\sigma_y = 0$$

$$(1)$$

In the investigation of thermal stresses, small dis-
placements will be assumed, such that effects of deformation
on equilibrium equation can be neglected.

In the equation (1), α and β are Lame' co-efficients
and they are given by the following equation.

$$\alpha = A\left(1 + z/r_1\right) \qquad\qquad \beta = B\left(1 + z/r_2\right) \qquad (2)$$

Figure (5.1) represents a differential element of a
shell, cut out by surfaces x=constant, and y=constant.
The variable x, y and z are orthogonal shell co-ordinates.
The tension (Nx,Ny), shears (Nxy, Nyx, Qx, Qy), twisting
moments (Mxy, Myx), and bending moments (Mx, My) per unit
length of the middle surface may be expressed in terms of
stress components($\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}$ and τ_{yz}).

The complete set of relations is,

$$N_x = \int_{-h/2}^{h/2} \sigma_x \left(1 + z/r_2\right) dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \left(1 + z/r_1\right) dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \left(1 + z/r_2\right) dz$$

$$N_{yx} = \int_{-h/2}^{h/2} \tau_{xy} \left(1 + z/r_1 \right) dz$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} \left(1 + z/r_2 \right) dz$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} \left(1 + z/r_1 \right) dz$$

$$M_x = \int_{-h/2}^{h/2} z \left(1 + z/r_2 \right) \sigma_x \, dz$$

$$M_y = \int_{-h/2}^{h/2} z \left(1 + z/r_1 \right) \sigma_y \, dz$$

$$M_{xy} = \int_{-h/2}^{h/2} z \left(1 + z/r_2 \right) \tau_{xy} \, dz$$

$$M_{yx} = \int_{-h/2}^{h/2} z \left(1 + z/r_1 \right) \tau_{xy} \, dz$$

(3)

These representations of tensions, shears etc. are similar to what Flügge (16) used.

The complete set of equations derived by Langhaar (14) is,

$$\frac{\partial}{\partial x}(BN_x) + \frac{\partial}{\partial y}(AN_{yx}) + N_{xy}\frac{\partial A}{\partial y} - N_y\frac{\partial B}{\partial x} + \frac{AB}{r_1}Q_x$$

$$+ ABP_x = 0$$

$$\frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial}{\partial y}(AN_y) + N_{yx}\frac{\partial B}{\partial x} - N_x\frac{\partial A}{\partial y} + \frac{AB}{r_2}Q_y$$

$$+ ABP_y = 0$$

$$\frac{\partial}{\partial x}(BQ_x) + \frac{\partial}{\partial y}(AQ_y) - \frac{AB}{r_1}N_x - \frac{AB}{r_2}N_y$$

$$+ ABP_z = 0$$

$$\frac{\partial}{\partial x}(BM_x) + \frac{\partial}{\partial y}(AM_{yx}) + M_{xy}\frac{\partial A}{\partial y} - M_y\frac{\partial B}{\partial x}$$

$$- ABQ_x + ABR_y = 0$$

$$\frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial}{\partial y}(AM_y) + M_{yx}\frac{\partial B}{\partial x} - M_x\frac{\partial A}{\partial y}$$

$$- ABQ_y - ABR_x = 0 \tag{4}$$

The equation (4) is supplemented by the identity,

$$\frac{M_{xy}}{r_1} - \frac{M_{yx}}{r_2} = N_{yx} - N_{xy} \tag{5}$$

V.2 Strain-Displacement Relations

Strain displacement relations for a shell has been discussed in Section IV. In this section we will assume those relations. However, for the problem of thermal stresses, the second order terms in the strain-displacement relations will be neglected.

V.3 Cylindrical Shell

Until now, the discussion made in this chapter is for general case. Now the particular case of orthotropic cylindrical shell will be considered.

For a cylindrical shell strain-displacement relations are given by equation (31) and (32), Section IV. If we neglect the second order terms, these relations become,

$$\varepsilon_x = \bar{u}_x - z\omega_{xx}$$

$$\varepsilon_y = (\bar{v}_y + \omega)/a - z\omega_{yy}/a^2$$

$$\gamma_{xy} = \bar{v}_x + \bar{u}_y/a - 2z\omega_{xy}/a \tag{6}$$

The stress-strain relations for orthotropic shell is given by,

$$\sigma_x = b_{11}\varepsilon_x + b_{12}\varepsilon_y - c_1\theta$$

$$\sigma_y = b_{12}\varepsilon_x + b_{22}\varepsilon_y - c_2\theta$$

$$\tau_{xy} = b_{66}\gamma_{xy} \tag{7}$$

Substituting (6) and (7) into (3) and noting that $r_1 = \infty$ and $r_2 = a$, we get,

$$N_x = h\left[b_{11}\bar{u}_x + b_{12}(\bar{v}_y + \omega)/a\right] - \frac{h^3}{12}\left[b_{12}\omega_{yy}/a^3 + \right.$$

$$\left. b_{11}\omega_{xx}/a\right] - \int_{-h/2}^{h/2} c_1\theta(1 + z/a)dz$$

$$N_y = h\left[b_{12}\bar{u}_x + b_{22}(\bar{v}_y + \omega)/a\right] - \int_{-h/2}^{h/2} c_2\theta \, dz$$

$$N_{xy} = h\left[b_{66}(\bar{v}_x + \bar{u}_y/a)\right] - \omega_{xy}h^3/6a^2$$

$$N_{yx} = h\left[b_{66}(\bar{v}_x + \bar{u}_y/a)\right]$$

$$M_x = \frac{h^3}{12a}\left[b_{11}\bar{u}_x + b_{12}\left(\frac{\bar{v}_y+\omega}{a} - \frac{\omega_{yy}}{a^2}\right) - b_{11}\omega_{xx}a\right]$$

$$- \int_{-h/2}^{h/2} C_1\theta\,(z+z^2/a)\,dz$$

$$M_y = -b_{12}\frac{h^3\omega_{xx}}{12} - \frac{b_{22}h^3}{12a^2}\omega_{yy} - \int_{-h/2}^{h/2} C_2 z\theta\,dz$$

$$M_{xy} = \frac{h^3}{12a}\left[b_{66}\left(\bar{v}_x + \bar{u}_y/a - 2\omega_{xy}\right)\right]$$

$$M_{yx} = -\frac{\omega_{xy}h^3}{6a} \tag{8}$$

For the case of axi-symmetrical loading and boundary conditions, Nx, Ny etc. take the following form.

$$N_x = h\left[b_{11}\bar{u}_x + b_{12}\,\omega/a\right] - \frac{h^3}{12a}b_{11}\,\omega_{xx} - \int_{-h/2}^{h/2} C_1\theta\,(1+z/a)\,dz$$

$$N_y = h\left[b_{12}\bar{u}_x + b_{22}\,\omega/a\right] - \int_{-h/2}^{h/2} C_2\theta\,dz$$

$$N_{xy} = 0$$

$N_{yx} = 0$

$$M_x = \frac{h^3}{12a} \left[b_{11} \bar{u}_x + b_{12} \omega/a - b_{11} \omega_{xx} a \right]$$

$$- \int_{-h/2}^{h/2} c_1 \theta (z + z^2/a) \, dz$$

$$M_y = - \frac{h^3 b_{12}}{12} \omega_{xx} - \int_{-h/2}^{h/2} c_2 z \theta \, dz$$

$M_{xy} = 0$

$M_{yx} = 0$

$$(9)$$

The equilibirium equation (4) for axi-symmetric case of cylindrical shell is reduced to

$$\frac{\partial N_x}{\partial x} = 0 \qquad\qquad \frac{\partial M_x}{\partial x} - Q_x = 0$$

$$\frac{\partial Q_x}{\partial x} - N_y/a + P_z = 0 \qquad\qquad (10)$$

From last two equations of (10) we get,

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{N_y}{a} + P_z = 0 \tag{11}$$

If the shell is considered to be homogeneous, then b_{11}, b_{12} etc. are independant of position. Now, substituting equation (9) into (10) and (11) following equations are obtained.

$$\frac{h^2}{12}\omega_{xxx} - a\bar{u}_{xx} - \frac{b_{12}}{b_{11}}\omega_x + \frac{a}{hb_{11}}\int_{-h/2}^{h/2}\frac{\partial}{\partial x}C_1\theta(1+z/a)dz = 0 \tag{11}$$

$$\frac{h^2}{12}\omega_{xxxx} + \frac{b_{22}}{b_{11}}\frac{\omega}{a^2} - \frac{h^2}{12a}U_{xxx} + \frac{b_{12}}{b_{11}}\frac{\bar{u}_x}{a} - \frac{P_z}{b_{11}h}$$

$$- \frac{b_{12}h^2}{b_{11}12a^2}\omega_{xx} - \frac{1}{ab_{11}h}\int_{-h/2}^{h/2}C_2\theta dz + \frac{1}{b_{11}h}\int_{-h/2}^{h/2}\frac{\partial^2}{\partial x^2}C_1\theta(z+z^2/a)dz = 0 \tag{12}$$

The equation (11) and (12) are the equilibrium equations of an orthotropic cylindrical shell in terms of its displacements when the loading is axi-symmetric. Solving these two equations u and w can be obtained and from there, thermal stresses can be computed. In the following pages few simple cases of temperature distributions and boundary conditions will be considered and the corresponding thermal stress problems will be solved.

For the thermal stress problem three diffferent temperature distributions will be considered - they are:

(a) Temperature varying along the thickness of the shell; $T = T(z)$

(b) Temperature varying along the generator $T = T(x)$

(c) Temperature varying along the generator and thickness; $T = T_1(x).T_2(z)$

Attention will be focussed only into simple supported case where the axial displacements are not prevented.

From equation (10), for axi-symmetrical deformation, we get,

$$\frac{\partial N_x}{\partial x} = 0$$

Therefore, Nx = constant

For simple supported case Nx at the end is zero
(since, there is no net axial force); thus Nx = 0.

Now, from equation (9) one obtains,

$$u_x = -\frac{b_{12}\,\omega}{a\,b_{11}} + \frac{h^2\omega_{xx}}{12a} + \frac{1}{h\,b_{11}} \int_{-h/2}^{h/2} C_1\,\theta\left(1+z/a\right)dz \quad (13)$$

Substituting the expression for u_x from the above
equation into equation (12) and considering no mechanical
load present, we get,

$$\omega_{xxxx} + 4\alpha^2\omega_{xx} + 4\beta^4\,\omega = f(\theta) \quad (14)$$

where,

$$\beta^4 = \frac{36}{h^2(12a^2-h^2)}\left[\frac{b_{22}}{b_{11}} - \frac{b_{12}^2}{b_{11}^2}\right] \quad (15)$$

$$\alpha^2 = \frac{3\,b_{12}}{b_{11}\left(12a^2-h^2\right)} \quad (16)$$

$$f(\theta) = \frac{144\,a^2}{h^2(12a^2 - h^2)} \left[\frac{1}{b_{11}h} \int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} C_1\theta\left(z + z^2/a\right)dz \right.$$

$$- \frac{h}{12ab_{11}} \int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} C_1\theta\left(1+z/a\right)dz + \frac{1}{h\,b_{11}}\left(\frac{b_{12}}{b_{11}a}\right)$$

$$\left. \times \int_{-h/2}^{h/2} C_1\theta\left(1+z/a\right)dz - \frac{1}{ab_{11}h}\int_{-h/2}^{h/2} C_2\theta\,dz \right. \qquad (17)$$

The general solution of the equation (14) is given by

$$\omega = e^{-\sqrt{\beta^2-\alpha^2}} \left[B_1 \cos\sqrt{\beta^2+\alpha^2}\,x + B_2 \sin\sqrt{\beta^2+\alpha^2}\,x \right]$$

$$+ e^{\sqrt{\beta^2-\alpha^2}} \left[B_3 \cos\sqrt{\beta^2+\alpha^2}\,x + B_4 \sin\sqrt{\beta^2+\alpha^2}\,x \right]$$

$$+ W_p \qquad (18)$$

Where B_1, B_2, B_3 and B_4 are arbitrary constants which can be evaluated from boundary conditions; W_p is the particular solution corresponding to the temperature function. A few temperature functions and related thermal stress problems have been considered in Section VII.

Equations (11) and (12) are the equilibrium equations
for orthotropic cylindrical shell, developed from general
equilibrium equations (1) given by Love (1). These equations
are rather difficult to handle, except for a few particular
cases of boundary conditions. Apart from this fact, since
the buckling criteria will be established using the strain
energy expression given by equation (28), Section IV, we
will develop simplified set of equilibrium equations, by
setting the first variation of strain energy equal to zero.
As before we will derive the equilibrium equations for
axi-symmetric case and without any mechanical loading.

For cylindrical shell, middle surface strain-displace-
ment relations in axi-symmetrical case can be approximated
by,

$$e_x = \bar{u}_x$$

$$e_y = \omega/a$$

$$\gamma_{xy} = 0 \tag{18}$$

Therefore membrane stresses will be given by,

$$\sigma_x = b_{11} \bar{u}_x + b_{12} \omega/a - C_1 \theta_m$$

$$\sigma_y = b_{12} \bar{u}_x + b_{22} \omega/a - C_2 \theta_m$$

$$\tau_{xy} = 0 \tag{19}$$

Where θ_m, temperature of the middle surface.

Considering the strain energy expression U given by equation (28), Section IV, the first variation δU, because of variation $\delta \bar{u}$ and $\delta \omega$ is given by,

$$\delta U = \iint \left\{ \left[b_{11} e_x + b_{12} e_y - C_1 \theta_0/h \right] \delta \bar{u}_x h \right.$$

$$+ h \left[b_{12} e_x + b_{22} e_y - C_2 \theta_0/h \right] \delta \omega/a + \left[C_1 \theta_1 \right.$$

$$\left. \left. - \frac{h^3}{12} b_{11} K_x \right] \delta \omega_{xx} \right\} a\,dx\,dy$$

$$(20)$$

If the shell segment is limited by x=a, x=b; y= φ_1 y= φ_2 the integration by parts yields,

$$\delta U = h \int_{\varphi_1}^{\varphi_2} \left[b_{11} e_x + b_{12} e_y - C_1 \theta_0/h \right] \delta \bar{u} \Big|_{x=a}^{x=b} a\,dy$$

$$- \int_{\varphi_1}^{\varphi_2} \left[\frac{h^3}{12} b_{11} K_x - C_1 \theta_1 \right] \delta \omega_x \Big|_{x=a}^{x=b} a\,dy$$

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$$+ \int_{\varphi_1}^{\varphi_2} \left[\frac{h^3}{12} b_{11} (K_x)_x - (C_1 \theta_1)_x \right] \delta\omega \Big|_{x=a}^{x=b} a \, dy$$

$$- \int_{\varphi_1}^{\varphi_2} \int_{a}^{b} \left[\frac{h^3}{12} b_{11} (K_x)_{xx} - (C_1 \theta_1)_{xx} \right] \delta\omega \, a \, dx \, dy$$

$$+ h \int_{\varphi_1}^{\varphi_2} \int_{a}^{b} \left[b_{11}(e_x)_x + (e_y)_x b_{12} - \left(\frac{C_1 \theta_0}{h} \right)_x \right] \delta\bar{u} \, a \, dx \, dy$$

$$+ h \int_{\varphi_1}^{\varphi_2} \int_{a}^{b} \left[b_{12} e_x + b_{22} e_y - C_2 \theta_0 / h \right] \frac{\delta\omega}{a} \, a \, dx \, dy \quad (21)$$

For the system to be in equilibrium, the first variation of potential energy must be zero. Therefore, both line integral and the surface integral must vanish. By equating the line integral to zero and replacing ~~e_x~~, ~~e_y~~ etc. by u, w and their derivatives, we obtain the following natural boundary conditions.

Along the circle x=a, x=b

$$\left[b_{11} \bar{u}_x + b_{12} \omega / a - C_1 \theta_0 / h \right] \delta\bar{u} = 0$$

$$\left[\frac{h^3}{12} b_{11} \omega_{xx} + C_1 \theta_1 \right] \delta\omega_x = 0$$

$$\left[\frac{h^3}{12} b_{11} \omega_{xxx} + (C_1 \theta_1)_x \right] \delta\omega = 0 \quad (22)$$

The equilibrium equations are obtained from the vanishing of surface integral and they are:

$$\frac{h^3}{12} b_{11} \omega_{xxxx} + h b_{12} \frac{\bar{u}_x}{a} + h b_{22} \frac{\omega}{a^2} - (C_1 \theta_1)_{xx}$$

$$+ C_2 \theta_0 / a = 0$$

$$b_{11} \bar{u}_{xx} + b_{12} \omega_x / a - (C_1 \theta_0)_x / h = 0 \qquad (23)$$

If we integrate the second equilibrium equation, we get,

$$a b_{11} \bar{u}_x + b_{12} \omega - a C_1 \theta_0 / h = \bar{K} \qquad (24)$$

(A) Simple Supported Case

Considering the first boundary condition of equation (21), $\bar{K}=0$ for simply supported case. Therefore, $\bar{u}_x$ can be written as

$$\bar{u}_x = - \frac{b_{12}}{a b_{11}} \omega + C_1 \theta_0 / h b_{11} \qquad (25)$$

Substituting this u_x in the first equilibrium equation (23), and rearranging we get,

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$$\frac{h^3}{12} b_{11} \omega_{xxxx} + \left[b_{22} - \frac{b_{12}^2}{b_{11}} \right] \frac{h\omega}{a^2} - (C_1 \theta_1)_{xx} - \frac{C_2 \theta_0}{a} + \frac{b_{12} C_1 \theta_0}{a b_{11}}$$

$$= 0 \qquad (26)$$

If C_1 is independant of space co-ordinate, then we can write the above equation in the following form

$$\omega_{xxxx} + 4K^4 \omega = A_1 (\theta_1)_{xx} + A_2 \theta_0 \qquad (27)$$

where,

$$4K^4 = \frac{12}{a^2 h^2 b_{11}} \left[b_{22} - \frac{b_{12}^2}{b_{11}} \right]$$

$$A_1 = \frac{12}{h^3 b_{11}} C_1$$

$$A_2 = \frac{12}{h^3 b_{11} a} \left(C_2 - \frac{b_{12}}{b_{11}} C_1 \right) \qquad (28)$$

General solution of equation (27) is given by

$$\omega = e^{-Kx} \left[B_1 \cos Kx + B_2 \sin Kx \right] + e^{Kx} \left[B_3 \cos Kx + B_4 \sin Kx \right] + W_p \qquad (29)$$

where, B_1, B_2, B_3 and B_4 are arbitrary constants to be evaluated from boundary conditions.

We note here, in both the cases where the temperature is function of x or z alone, the term $A(\theta_1)_{xx}$ vanishes. Therefore equation reduces to

$$\omega_{xxxx} + 4K^4\omega = A_2\theta_0 \qquad (30)$$

(B)   Fixed End Case:

For fixed end case the following condition is to be satisfied,

$$\int_0^\ell e_x\,dx = \int_0^\ell \bar{u}_x\,dx = 0 \qquad (31)$$

From the equation (24) we have

$$ab_{11}\bar{u}_x + b_{12}\omega - ac_1\theta_0/h = \bar{K}$$

Where $\bar{K}$ is a constant and can be evaluated from the boundary condition (31). Substituting the value of $\bar{u}_x$ from equation (24) into (23), we get,

$$\omega_{xxxx} + 4K^4\omega = A_1(\theta_1)_{xx} + A_2\theta_0 - \frac{\bar{K}\,12\,b_{12}}{a^2\,b_{11}^2\,h^2} \qquad (32)$$

where K, A_1 and A_2 are given by the equation (28). If the temperature distribution is a function of x or z alone, the equation (32) reduces to,

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$$\omega_{xxxx} + 4K^4\omega = A_2\Theta_o - \frac{\overline{K}.12.b_{12}}{a^2 b_{11}^2 h^2} \tag{33}$$

Particular cases of temperature distribution and related thermal stress problem will be discussed in Section VII.

V.4 Thermal Stresses in a Conical Shell

For a conical shell strain-displacement relations, as given by equation (36) in Section IV, are as follows:

$$\varepsilon_x = \overline{u}_x - z\,\omega_{xx}$$

$$\varepsilon_y = \frac{\overline{v}_y}{x\sin\alpha} + \frac{\overline{u}}{x} + \frac{\omega}{x\tan\alpha} - \frac{\omega_x}{x}z - \frac{\omega_{yy}}{x^2\sin^2\alpha}z$$

$$\gamma_{xy} = \overline{v}_x + \frac{\overline{u}_y}{x\sin\alpha} - \frac{\overline{v}}{x} - \frac{2z\,\omega_y}{x^2\sin\alpha} - \frac{2\omega_{xy}}{x\sin\alpha}z \tag{34}$$

Substituting these strains in stress-strain relation (7) and considering an axi-symmetrical case, we get,

$$\sigma_x = b_{11}(\overline{u}_x - z\,\omega_{xx}) + b_{12}\left(\frac{\overline{u}}{x} - \frac{\omega}{x\tan\alpha} - \frac{\omega_x}{x}z\right) - C_1\theta$$

$$\sigma_y = b_{12}(\overline{u}_x - z\,\omega_{xx}) + b_{22}\left(\frac{\overline{u}}{x} - \frac{\omega}{x\tan\alpha} - \frac{\omega_x}{x}z\right) - C_2\theta$$

$$\tau_{xy} = 0 \tag{35}$$

Substituting equation (35) into (3) and noting that $r_1 = \infty$ and $r_2 = x\tan\alpha$, we get,

$$N_x = h\left[b_{11}\bar{u}_x + b_{12}\frac{\bar{u}}{x} + b_{12}\frac{\omega}{x\tan\alpha}\right] - \frac{h^3}{12x\tan\alpha}\left(b_{11}\omega_{xx}\right.$$

$$\left. - b_{12}\omega x/x\right) - \int_{-h/2}^{h/2}C_1\theta\left(1+\frac{z}{x\tan\alpha}\right)dz$$

$$N_y = h\left[b_{11}\bar{u}_x + b_{22}\frac{\bar{u}}{x} + b_{22}\frac{\omega}{x\tan\alpha}\right] - \int_{-h/2}^{h/2}C_2\theta\,dz$$

$$N_{xy} = N_{yx} = M_{xy} = M_{yx} = 0$$

$$M_x = \frac{h^3}{12x\tan\alpha}\left[b_{11}\bar{u}_x + b_{12}\frac{\bar{u}}{x} + b_{12}\frac{\omega}{x\tan\alpha}\right]$$

$$- \left[b_{11}\omega_{xx} + b_{12}\omega x/x\right]\frac{h^3}{12} - \int_{-h/2}^{h/2}C_1\theta\left(1+\frac{z}{x\tan\alpha}\right)dz$$

$$M_y = -\frac{h^3}{12}\left(b_{12}\omega_{xx} + b_{22}\omega x/x\right) - \int_{-h/2}^{h/2}C_2\theta z\,dz \tag{36}$$

For conical shell equation (4) reduces to,

$$\frac{\partial}{\partial x}(x N_x) - N_y = 0 \tag{37}$$

$$\frac{\partial}{\partial x}(x Q_x) - \frac{N_y}{\tan\alpha} = 0 \tag{38}$$

$$\frac{\partial}{\partial x}(x M_x) - M_y - x Q_x = 0 \tag{39}$$

From (38) to (39), one obtains

$$\frac{\partial^2}{\partial x^2}(x M_x) - \frac{\partial}{\partial x} M_y - \frac{N_y}{\tan\alpha} = 0 \tag{40}$$

Substituting the values of Mx, My, Ny and Nx in the equation (37) and (40), we get the following differential equations:

$$\left[-\frac{b_{11} h^3}{12} x \right] \omega_{xxxx} - \left[\frac{2 b_{11} h^3}{12} \right] \omega_{xxx} + \left[\frac{b_{22} h^3}{12 x} + \right.$$

$$\left. \frac{b_{12} h^3}{12 x \tan^2\alpha} \right] \omega_{xx} - \left[\frac{2 b_{12} h^3}{12 x^2 \tan^2\alpha} + \frac{b_{22} h^3}{12 x^2} \right] \omega_x +$$

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$$\bar{u}_{xxx}\left[\frac{b_{11}h^3}{12\tan\alpha}\right] + \bar{u}_{xx}\left[\frac{h^3b_{12}}{12x\tan\alpha}\right] - \bar{u}_x\left[\frac{2b_{12}h^3}{12x^2\tan\alpha}\right.$$

$$\left. + \frac{b_{12}h}{\tan\alpha}\right] + \bar{u}\left[\frac{2b_{12}h^3}{12x^3\tan\alpha} - \frac{b_{22}h}{x\tan\alpha}\right] + F_1(\theta) = 0 \tag{41}$$

$$[hb_{11}x]\bar{u}_{xx} + [hb_{11}]\bar{u}_x - \left[\frac{hb_{22}}{x}\right]\bar{u} - \left[\frac{h^3b_{11}}{12\tan\alpha}\right]\omega_{xxx}$$

$$+ \left[\frac{h^3b_{12}}{12x\tan\alpha}\right]\omega_{xx} - \left[\frac{h^3b_{12}}{12x^2\tan\alpha}\right]\omega_x - \left[\frac{hb_{22}}{x\tan\alpha}\right]\omega + F_2(\theta) = 0 \tag{42}$$

where,

$$F_1(\theta) = -\frac{\partial^2}{\partial x^2}\int_{-h/2}^{h/2} C_1 x\theta z\left(1 + z/x\tan\alpha\right)dz$$

$$+ \int_{-h/2}^{h/2} C_2\theta z\,dz$$

$$+ \int_{-h/2}^{h/2} \frac{C_2\theta}{\tan\alpha}\,dz \tag{43}$$

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$$F_2(\theta) = -\frac{\partial}{\partial x} \int_{-h/2}^{h/2} x\, C_1\, \theta \left(1 + z/x \tan\alpha \right) dz + \int_{-h/2}^{h/2} C_2\, \theta\, dz \qquad (44)$$

Equations (41) and (42) are frightfully complicated and no effort will be made to solve them. In the following pages simpler differential equations defining displacements of conical shell will be developed. Instead of deriving differential equation for axi-symmetric case, a general case of deformation will be considered with arbitrary load and temperature distribution. The equation will be derived following Hoff's (18) work. Hoff derived the equation for isotropic case subjected to mechanical loading. In the present case thermal loading will be also present and shell material will be orthotropic.

Derivation of Simplified Differential Equation for Conical Shell

The present derivation will be restricted to small cone angle and truncated cone.

Considering small cone angle the membrane strain-displacement relations can be written as,

$$e_x = \bar{u}_x$$

$$e_y = (\bar{v}_y + \omega \cos\alpha)/x\sin\alpha$$

$$e_{xy} = \bar{v}_x + \bar{u}_y/x\sin\alpha \tag{45}$$

The curvature can be written as

$$K_x = -\omega_{xx}$$

$$K_y = -\omega_{yy}/x^2\sin^2\alpha$$

$$K_{xy} = -2\omega_{xy}/x\sin\alpha \tag{46}$$

We note equations (45) and (46) differ considerably from other previous strain-displacement and curvature-displacement relations. But the effect of additional terms are appreciable when cone angle is not small.

The middle surface stresses will be given by,

$$\sigma_x = b_{11}\bar{u}_x + b_{12}(\bar{v}_y + \omega\cos\alpha)/x\sin\alpha - c_1\theta$$

$$\sigma_y = b_{12}\bar{u}_x + b_{22}(\bar{v}_y + \omega\cos\alpha)/x\sin\alpha - c_2\theta$$

$$\tau_{xy} = b_{66}(\bar{v}_x + \bar{u}_y/x\sin\alpha) \tag{47}$$

The total potential energy of an orthotropic shell is given by

$$V = U + \Omega \qquad (48)$$

where U is the strain energy and Ω is potential energy due to external loading.

The strain energy expression as given by equation (28), Section IV, can be separated into three parts U_m, U_b, U_θ, where,

$$U_m = \frac{1}{2} \iint \left[b_{11} e_x^2 + b_{22} e_y^2 + b_{66} e_{xy}^2 + 2 b_{12} e_x e_y \right] h \, AB \, dx \, dy$$

$$U_b = \frac{1}{24} \iint \left[b_{11} K_x^2 + b_{22} K_y^2 + b_{66} K_{xy}^2 + 2 b_{12} K_x K_y \right] h^3 AB \, dx \, dy$$

$$U_\theta = - \iint \left[(C_1 e_x + C_2 e_y) \theta_0 + (C_1 K_x + C_2 K_y) \theta_1 \right] AB \, dx \, dy$$

$$(49)$$

For a cone, $A = 1$ and $B = x \sin\alpha$

The condition of equilibrium will be established by setting the first variation of the total potential energy equal to zero. In the derivation it will be assumed that the temperature distribution is independant of radial co-ordinate and therefore, $\theta_1 = 0$.

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The change in membrane and thermal energy $\delta(U_m + U_\theta)$ caused by variation $\delta\bar{u}$, $\delta\bar{v}$ and δw can be written as,

$$\delta(U_m + U_\theta) = h \iint \left[\sigma_x\, \delta\bar{u}_x + (\sigma_y/x\sin\alpha)\, \delta\bar{v}_y \right.$$

$$+ (\sigma_y/x\sin\alpha)\cos\alpha\, \delta\omega + \tau_{xy}\, \delta\bar{v}_x$$

$$\left. + (\tau_{xy}/x\sin\alpha)\, \delta\bar{u}_y \right] x\sin\alpha\, dx\, dy \qquad (50)$$

If the shell segment is limited by the line x=a, x=b and, y=φ_1, y=φ_2 then integration by parts yields,

$$\delta(U_m + U_\theta) = h\sin\alpha \int_{\varphi_1}^{\varphi_2} x\,(\sigma_x\,\delta\bar{u} + \tau_{xy}\,\delta\bar{v})\Big|_{x=a}^{x=b} dy$$

$$+ h \int_a^b (\sigma_y\,\delta\bar{v} + \tau_{xy}\,\delta\bar{u})\Big|_{y=\varphi_1}^{y=\varphi_2} dx\ -$$

$$- h \int_a^b \int_{\varphi_1}^{\varphi_2} \left\{ \left[(x\sin\alpha\,\sigma_x)_x + (\tau_{xy})_y \right] \delta\bar{u} + \left[(\sigma_y)_y \right.\right.$$

$$+ (x\sin\alpha\,\tau_{xy})_x \Big]\, \delta\bar{v} + \sigma_y\cos\alpha\,\delta\omega \Big\}\, dy\, dx$$

Similarly, for δU_b one obtains,

$$\delta U_b = \frac{h^3}{12} \left\{ \int_{\varphi_1}^{\varphi_2} \left\{ \left[x\sin\alpha\, b_{11}\, \omega_{xx} + \frac{b_{12}}{x\sin\alpha}\, \omega_{yy} \right] \delta\omega_x \right\}_{x=a}^{x=b} dy \right.$$

$$- \int_{\varphi_1}^{\varphi_2} \left\{ \left\{ (x\sin\alpha\, b_{11}\, \omega_{xx})_x + \left(\frac{b_{12}}{x\sin\alpha}\, \omega_{yy} \right)_x + \frac{4b_{66}}{x\sin\alpha}\, \omega_{xyy} \right\} \delta\omega \right\}_{x=a}^{x=b} dy$$

$$+ \int_{a}^{b} \left\{ \left\{ \left[\frac{b_{22}}{x^3\sin^3\alpha}\, \omega_{yy} + \frac{b_{12}\,\omega_{xx}}{x\sin\alpha} \right] \delta\omega_y \right\}_{y=\varphi_1}^{y=\varphi_2} dx \right.$$

$$- \int_{a}^{b} \left\{ \left\{ \frac{b_{22}}{x^3\sin^3\alpha}\, \omega_{yyy} + \frac{b_{12}}{x\sin\alpha}\, \omega_{xxy} \right\} \delta\omega \right\}_{y=\varphi_1}^{y=\varphi_2} dx$$

$$- \int_{a}^{b} \left\{ \frac{4b_{66}}{x\sin\alpha}\, \omega_{xy}\, \delta\omega_x \right\}_{y=\varphi_1}^{y=\varphi_2} dx$$

$$+ \int_{a}^{b} \int_{\varphi_1}^{\varphi_2} \left\{ (x\sin\alpha\, b_{11}\, \omega_{xx})_{xx} + \left(\frac{b_{12}}{x\sin\alpha}\, \omega_{yy} \right)_{xx} \right.$$

$$\left. + \frac{4b_{66}}{x\sin\alpha}\, \omega_{xxyy} + \frac{b_{22}}{x^3\sin^3\alpha}\, \omega_{yyyy} + \frac{b_{12}}{x\sin\alpha}\, \omega_{xxyy} \right\} \delta\omega\, dx\, dy \Bigg\}$$

$$(52)$$

To find the $\delta\Omega$, let us consider that the surface loads X, Y, Z force per unit area, are acting in axial, circumferential and radial directions.

Therefore the change in potential energy $\delta\Omega$ of these loads during virtual displacements is

$$\delta\Omega = -\int_a^b \int_{\varphi_1}^{\varphi_2} \left(X\,\delta\bar{u} + Y\,\delta\bar{v} + Z\,\delta\omega \right) x\sin\alpha \, dy\, dx \qquad (53)$$

For the system to be in equilibrium, the first variation of potential energy must be zero. One of the requirements for the equilibrium is, therefore, the vanishing of the line integrals. From there we obtain the following natural boundary conditions.

Along the circle x=a and x=b

$$\left\{ b_{11}\,\bar{u}_x + b_{12}\,(\bar{v}_y + \omega\cos\alpha)/x\sin\alpha - c_1\theta \right\} \delta\bar{u} = 0$$

$$\left\{ \bar{v}_x + \bar{u}_y / x\sin\alpha \right\} \delta\bar{v} = 0$$

$$\left\{ x\sin\alpha\, b_{11}\,\omega_{xx} + b_{12}\,\omega_{yy}/x\sin\alpha \right\} \delta\omega_x = 0$$

$$\left\{ (x\sin\alpha\, b_{11}\,\omega_{xx})_x + \left(\frac{b_{12}}{x\sin\alpha}\,\omega_{yy}\right)_x + \frac{4b_{66}}{x\sin\alpha}\,\omega_{xyy} \right\} \delta\omega = 0$$
$$(54a)$$

Along the generator $y = \varphi_1$ and $y = \varphi_2$

$$\left\{ b_{12} \, \bar{u}_x + b_{22} \left(\bar{v}_y + \omega \cos\alpha \right) / x \sin\alpha - C_2 \theta \right\} \delta \bar{v} = 0$$

$$\left\{ \bar{v}_x + \bar{u}_y / x \sin\alpha \right\} \delta \bar{u} = 0$$

$$\left\{ \frac{b_{22}}{x^3 \sin^3\alpha} \omega_{yyy} + \frac{b_{12} \, \omega_{xxy}}{x \sin\alpha} + 4 b_{66} \left(\omega_{xy} / x \sin\alpha \right)_x \right\} \delta\omega = 0$$

$$\left\{ \frac{b_{22} \, \omega_{yy}}{x^3 \sin^3\alpha} + \frac{b_{12} \, \omega_{xx}}{x \sin\alpha} \right\} \delta\omega_y = 0 \tag{54 b}$$

The second condition for equilibrium will be that the sum of terms under the surface integral which are multiplied by $\delta(u,v,\omega)$, must vanish separately. From there, the following three equilibrium equations are obtained.

$$\left(x \sin\alpha \, \sigma_x \right)_x + \left(\tau_{xy} \right)_y + x \sin\alpha \left(X / h \right) = 0$$

$$\left(\sigma_y \right)_y + \left(x \sin\alpha \, \tau_{xy} \right)_x + x \sin\alpha \left(Y / h \right) = 0$$

$$\left(b_{11} \, x \sin\alpha \, \omega_{xx} \right)_{xx} + \left(\frac{b_{12} + 4 b_{66}}{x \sin\alpha} \right) \omega_{xxyy} + \left(\frac{b_{12}}{x \sin\alpha} \omega_{yy} \right)_{xx}$$

$$+ \frac{b_{22}}{x^3 \sin^3\alpha} \omega_{yyyy} - \frac{12}{h^2} \sigma_y \cos\alpha - \left(12 Z / h^3 \right) x \sin\alpha = 0 \tag{55}$$

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The last of equation (55) can be written as,

$$G_2(\omega) - \frac{12}{h^2} \cos\alpha \, x\sin\alpha \, \sigma_y - \left(\frac{12\,z}{h^3}\right) x^2 \sin^2\alpha = 0 \tag{56}$$

where,

$$G_2(\omega) = x\sin\alpha \left(x\sin\alpha \, b_{11} \, \omega_{xx}\right)_{xx} + (b_{12} + 4\,b_{66}) \omega_{xxyy}$$

$$+ x\sin\alpha \left(\frac{b_{12}\,\omega_{yy}}{x\sin\alpha}\right)_{xx} + \frac{b_{22}}{x^2\sin^2\alpha} \omega_{yyyy} \tag{57}$$

Substituting the stress-displacement relations into the first of two equilibrium equations, we get,

$$\left\{ x\sin\alpha \left[b_{11} \, \bar{u}_x + b_{12} \, (\bar{v}_y + \omega\cos\alpha)/x\sin\alpha - C_1\theta \right] \right\}_x$$

$$+ \left[b_{66} \left(\bar{v}_x + \bar{u}_y/x\sin\alpha \right) \right]_y + x\sin\alpha \left(X/h\right) = 0 \tag{58}$$

$$\left\{ b_{12} \, \bar{u}_x + b_{22} \, (\bar{v}_y + \omega\cos\alpha)/x\sin\alpha - C_2\theta \right\}_y$$

$$+ \left[b_{66} \left(\bar{v}_x + \bar{u}_y/x\sin\alpha \right) \right]_x + x\sin\alpha \left(Y/h\right) = 0 \tag{59}$$

From equation (59)

$$\bar{u}_{xy} = -\frac{1}{(b_{12}+b_{66})}\left[\frac{b_{22}}{x\sin\alpha}\bar{v}_{yy} + (x\sin\alpha\, b_{66}\,\bar{v}_x)_x\right.$$

$$\left. - C_2\theta_y + \frac{b_{22}\cos\alpha}{x\sin\alpha}\omega_y + x\sin\alpha\left(Y/h\right)\right] \qquad (60)$$

From equation (58)

$$\bar{v}_{xy} = -\frac{1}{(b_{12}+b_{66})}\left\{b_{11}(x\sin\alpha\,\bar{u}_x)_x + b_{12}\cos\alpha\,\omega_x\right.$$

$$\left. - (x\sin\alpha\, C_1\theta)_x + b_{66}\left[\frac{\bar{u}_y}{x\sin\alpha}\right]_y + x\sin\alpha\, X/h\right\}$$

$$(61)$$

Equations (58) and (59) are first differentiated with respect to y, multiplied by xsin and then differentiated with respect to x. In the expressions thus obtained, the values of $\bar{u}_{xy}$ and $\bar{v}_{xy}$ are substituted from equations (60) and (61). After re-arrangement these equations can be written as,

$$G_1(v) = \cos\alpha\left[-\frac{b_{22}b_{11}}{(b_{12}+b_{66})}(x\sin\alpha\,\omega_{xy})_x\,(x\sin\alpha)\right.$$

$$- \frac{b_{66}b_{22}}{(b_{12}+b_{66})} \omega_{yyy} + b_{12}(x\sin\alpha\,\omega_{xy})_x (x\sin\alpha) \Big]$$

$$- \frac{x\sin\alpha\,b_{11}}{(b_{12}+b_{66})} \Big[x\sin\alpha\,(x^2\sin^2\alpha\,Y/h)_x \Big]_x + \frac{b_{66}C_2\theta_{yyy}}{(b_{12}+b_{66})}$$

$$+ x^2\sin^2\alpha\,(Y/h)_{yy} + (x^2\sin^2\alpha\,X_y)_x (x\sin\alpha)/h$$

$$+ x\sin\alpha\Big[C_1 + \frac{b_{11}C_2}{(b_{12}+b_{66})} \Big] \Big[x\sin\alpha\,(x\sin\alpha\,\theta_y)_x \Big]_x$$

$$\tag{62}$$

and,

$$G_1(\bar{u}) = -\cos\alpha \Big\{ \frac{b_{12}b_{66}}{(b_{12}+b_{66})} x\sin\alpha\,[x\sin\alpha\,(x\sin\alpha\,\omega_x)_x]_x$$

$$- b_{22}x\sin\alpha\,\omega_{yyx} \Big\} - \frac{b_{12}b_{22}}{b_{66}+b_{11}}\cos\alpha\,x\sin\alpha\,\omega_{xyy}$$

$$+ \Big(\frac{b_{22}C_1}{b_{12}+b_{66}} - C_2 \Big)(x\sin\alpha\,C_1\theta_{yy})_x(x\sin\alpha) - \frac{b_{22}}{(b_{66}+b_{12})} \times$$

$$\Big(x^2\sin^2\alpha\,\frac{X_{yy}}{h} \Big) + \Big\{ \frac{x\sin\alpha\,b_{66}}{(b_{66}+b_{12})}(x\sin\alpha\,C_1\theta)_x x\sin\alpha \Big\}_x x\sin\alpha$$

$$+ \Big(\frac{x^2\sin^2\alpha}{h}Y_y \Big)_x x\sin\alpha - \frac{b_{66}\,x\sin\alpha}{b_{66}+b_{12}}\Big[x\sin\alpha\,(\frac{x^2\sin^2\alpha}{h}X)_x \Big]_x$$

$$\tag{63}$$

where, the operator $G_1(z)$ will be defined as,

$$G_1(z) = x \sin\alpha \frac{b_{11} b_{66}}{b_{12} + b_{66}} \left\{ x\sin\alpha \left[x\sin\alpha \left(x\sin\alpha Z_x \right)_x \right]_x \right\}_x$$

$$+ x\sin\alpha \left[\frac{b_{66}^2 + b_{11} b_{22}}{b_{12} + b_{66}} - (b_{12} + b_{66}) \right] \left(x\sin\alpha Z_{yyx} \right)_x$$

$$+ \frac{b_{66} b_{22}}{b_{12} + b_{66}} Z_{yyyy} \tag{64}$$

Now if the operator G_1 is applied to each term of equation (56), then with the help of equations (62) and (63), one arrives at the following differential equation,

$$G_1(G_2(\omega)) - \frac{12x}{h^2} \cos^2\alpha \sin\alpha \left(\frac{b_{11} b_{22} - b_{12}^2}{(b_{12} + b_{66})} \right) b_{66} \left\{ x\sin\alpha \left[x\sin\alpha \right. \right.$$

$$\left. \left. (x\sin\alpha \, \omega x)_x \right]_x \right\}_x + \frac{12}{h^3} \cos\alpha \left\{ \frac{b_{66}}{b_{66} + b_{12}} x\sin\alpha \left\{ x\sin\alpha \right. \right.$$

$$\left[x\sin\alpha \left(x^2\sin^2\alpha \; X_{yy} \right)_x + \left(\frac{b_{22} b_{11}}{b_{66} + b_{12}} - b_{12} \right) \left[x\sin\alpha \left(x^2\sin^2\alpha \right. \right. \right.$$

$$\left. \left. \left. X_y \right)_x \right]_x - b_{22} \, x^2\sin^2\alpha \; Y_{yyy} \right\} - \frac{12}{h^3} H_1 \left[Z \, x^2\sin^2\alpha \right]$$

$$-\frac{12}{h^3} \cos\alpha \left\{ \frac{C_1 b_{12} b_{66}}{(b_{12}+b_{66})} x \sin\alpha \left\{ x\sin\alpha \left\{ x\sin\alpha \right. \right. \right.$$

$$\left. \left. \times \left[x\sin\alpha (x\sin\alpha)_x \right]_x \right\}_x \right\}_x + x\sin\alpha \left\{ C_2 \left[\frac{b_{11} b_{22}}{(b_{12}+b_{66})} - b_{12} \right] \right.$$

$$\left. + C_1 \left[\frac{b_{22} b_{12}}{(b_{66}+b_{11})} - b_{22} \right] \right\} \times \left\{ (x\sin\alpha\, \theta_{yy})_x \, x\sin\alpha \right\}_x$$

$$+ \frac{b_{66} b_{22}}{(b_{12}+b_{66})} C_2 \theta_{yyyy} \Bigg\}$$

$$+ \frac{12}{h^2} H_1 (C_2\, \theta\, x\sin\alpha) \qquad\qquad (65)$$

Equation (65) is a differential equation in w alone. This equation may be solved for w and the expression for w thus obtained may be substituted into the equations (63) and (64) to get $\bar{u}$ and $\bar{v}$.

For axi-symmetrical temperature distribution without any mechanical load, equations (64) and (65) becomes,

$$G_1(\bar{u}) = -\frac{b_{12} b_{66}}{(b_{12}+b_{66})} \cos\alpha\, x\sin\alpha \left[x\sin\alpha (x\sin\alpha\, \omega_x)_x \right]_x$$

$$+ \left\{ \left[\frac{x\sin\alpha\, b_{66}}{(b_{66}+b_{12})} (x\sin\alpha\, C_1 \theta)_x \right]_x x\sin\alpha \right\}_x x\sin\alpha \qquad (66)$$

and

$$G_1(G_2(\omega)) - \frac{12x}{h^2} \sin\alpha \cos^2\alpha \frac{(b_{11}b_{22}-b_{12}^2)b_{66}}{(b_{12}+b_{66})} \times \left\{ x\sin\alpha \times \right.$$

$$\left. \left[x\sin\alpha \, (x\sin\alpha \, \omega_x)_x \right]_x \right\}_x + \frac{12b_{66}}{(b_{66}+b_{12})h^2} \cos\alpha \left[C_2 b_{11} - C_1 b_{12} \right] \times$$

$$x\sin\alpha \left\{ x\sin\alpha \left\{ x\sin\alpha \left[x\sin\alpha \, (x\sin\alpha \, \theta)_x \right]_x \right\}_x \right\}_x$$

$$(67)$$

where $G_1(z)$ and $G_2(z)$ have now been reduced to,

$$G_1(z) = x\sin\alpha \frac{b_{11}b_{66}}{(b_{12}+b_{66})} \left\{ x\sin\alpha \left[x\sin\alpha \, (x\sin\alpha \, z_x)_x \right]_x \right\}_x \quad (68)$$

$$G_2(z) = b_{11} \, x\sin^2\alpha \, (xz_{xx})_{xx} \qquad (69)$$

For isotropic case the equation (67) reduces that given by Hoff and Singer (19), except for $G_2(z)$. The reason for that is a more accurate expression for curvature they used in their work.

Equations (66) and (67) can be simplified and written as

$$b_{11} (x\Box_x)_x = -b_{12} \cot\alpha \, \omega_x + C_1 (x\theta)_x \qquad (70)$$

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and

$$G_2(\omega) - \frac{12}{h^2} \cos^2\alpha \left(\frac{b_{22} b_{11} - b_{12}^2}{b_{11}} \right) \omega$$

$$+ \frac{12}{h^2} \cos\alpha \sin\alpha \left[\frac{c_1 b_{11} - c_2 b_{12}}{b_{11}} \right] x\theta = 0 \tag{71}$$

Equation (71) can be written in the following form,

$$x^2 \omega_{xxxx} + 2x \omega_{xxx} - B'^4 \omega = -c' x\theta \tag{72}$$

where,

$$B'^4 = \frac{12}{h^2} \cos^2\alpha \frac{\left(b_{22} b_{11} - b_{12}^2 \right)}{b_{11}^2 \sin^2\alpha}$$

$$c' = \frac{12}{h^2} \cos\alpha \left(c_1 b_{11} - c_2 b_{12} \right) \sin\alpha / b_{11}^2 \tag{73}$$

It may be convenient at this stage to use non-dimensional form of diplacements and distance (See Figure 5.2).

Let,

$$\bar{x} = x/a$$

$$\bar{u} = \bar{u}/a$$

$$\bar{\omega} = \omega/a$$

Then equation (72) can be written as,

$$\bar{x}^2 \omega_{\bar{x}\bar{x}\bar{x}\bar{x}} + 2\bar{x}\omega_{\bar{x}\bar{x}\bar{x}} - B^4 \bar{\omega} = C\bar{x}\theta \qquad (74)$$

where

$$B^4 = a^2 B'^4$$

$$C = a^2 c' \qquad (75)$$

For homogeneous part of the equation (74), an approximate solution in the form of asymptotic expansion may be obtained by the procedure suggested by Love(1).

Let,

$$\bar{\omega}_c = e^{\sqrt{z}} \left(z^{1/4} + a z^{-1/4} + b z^{-3/4} + c z^{-5/4} + \cdots \cdots \right) \qquad (76)$$

where,

$$z = m^2 \bar{x}$$

Substituting this equation (74) and setting right hand side of the equation equals to zero, we get.

$$m^4 e^{\sqrt{z}} \left\{ z^{1/4} + (a-4) z^{-1/4} + (b - 8a - 9/2) z^{-3/4} + \right.$$

$$\left(c - 12b + \frac{113a}{4} + 6 \right) z^{-5/4} + \left(-16c + \frac{129}{2}b - 51a - \frac{231}{16} \right) z^{-7/4} + \cdots$$

$$+ 4z^{-1/4} + (4a - 6)z^{-3/4} + (4b - 18a - 13)z^{-5/4}$$

$$+ (4c - 30b - 33a + 21/2)z^{-7/4} + \cdots \Big\}$$

$$- B^4 e^{\sqrt{z}} \Big(z^{1/4} + az^{-1/4} + bz^{-3/4} + cz^{-5/4} + \cdots \Big) = 0 \qquad (78)$$

Equating co-efficients of $z^{-1/4}$ to zero, one obtains,

$$m^4 - 16 B^4 = 0$$

$$m_{1,\cdots 4} = 2B, -2B, 2Bi, -2Bi$$

Equating the coefficients of $z^{-3/4}$, $z^{-5/4}$, $z^{-7/4}$ respectively to zero, we get

$$a = -3/8$$

$$b = (41a/4 - 7)/12$$

$$c = 69b/2 - 84a - 231/16 \qquad (79)$$

For a thin shell three or four terms will be good enough. The complementary solution of the equation after little manipulation can be written in the form,

$$\omega_c = R_1 L_1 + R_2 L_2 + R_3 L_3 + R_4 L_4$$

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where,

$$L_1 = e^{2B\sqrt{\bar{x}}} \left[\bar{x}^{1/4} + \frac{a}{2B} \bar{x}^{-1/4} + \frac{b}{(2B)^2} \bar{x}^{-3/4} + \frac{c}{(2B)^3} \bar{x}^{-5/4} + \cdots \right]$$

$$L_2 = e^{-2B\sqrt{\bar{x}}} \left[\bar{x}^{1/4} - \frac{a}{2B} \bar{x}^{-1/4} + \frac{b}{(2B)^2} \bar{x}^{-3/4} - \frac{c}{(2B)^3} \bar{x}^{-5/4} + \cdots \right]$$

$$L_3 = \bar{x}^{1/4} \cos\left(2B\sqrt{\bar{x}} + \pi/4\right) + \frac{a\bar{x}^{-1/4}}{2B} \cos\left(2B\sqrt{\bar{x}} - \pi/4\right)$$

$$+ \frac{b\bar{x}^{-3/4}}{(2B)^2} \cos\left(2B\sqrt{\bar{x}} - 3/4\pi\right) + \frac{c\bar{x}^{-5/4}}{(2B)^3} \cos\left(2B\sqrt{\bar{x}} - 5/4\pi\right)$$
$$+ \cdots$$

$$L_4 = \bar{x}^{1/4} \sin\left(2B\sqrt{\bar{x}} + \pi/4\right) + \frac{a\bar{x}^{-1/4}}{2B} \sin\left(2B\sqrt{\bar{x}} - \pi/4\right)$$

$$+ \frac{b\bar{x}^{-3/4}}{(2B)^2} \sin\left(2B\sqrt{\bar{x}} - 3/4\pi\right) + \frac{c\bar{x}^{-5/4}}{(2B)^3} \cos\left(2B\sqrt{\bar{x}} - 5/4\pi\right)$$
$$+ \cdots$$

$$(81)$$

For the particular integral of equation (74), we note that if the temperature distribution can be represented by a polynomial, the particular integral may be obtained

easily by the method of undetermined co-efficients.

The arbitrary constants R_1, R_2, R_3 and R_4 . can be evaluated from boundary conditions. Once w is determined, $\bar{u}$ can be determined from equation (70). For simple supported case equation (70) can be written as,

$$\bar{x}\,\bar{u}_{\bar{x}} = -\frac{b_{12}}{b_{11}}\cot\alpha\,\bar{\omega} + \frac{C_1(\bar{x}\theta)}{b_{11}} + G \qquad (82)$$

From boundary condition (54a) in case of simple support where there is no axial restraints,

$$G = O \qquad\qquad (83)$$

The different stress can then be written as,

$$\sigma_x = O$$

$$\sigma_y = \frac{\bar{\omega}}{\bar{x}}\cot\alpha\left(b_{22} - \frac{b_{12}^2}{b_{11}}\right) + \theta\left[\frac{b_{12}}{b_{11}}C_1 - C_2\right]$$

$$\tau_{xy} = O$$

$$(84)$$

We note $\sigma_x = 0$ is a consequence of approximate nature of middle surface strain assumed.

For the fixed end case, the boundary conditions will be,

$$\left.\begin{array}{l} \omega = 0 \\ \omega_x = 0 \end{array}\right\} \quad \text{at} \quad \bar{x} = 1 \;,\; 1+\bar{l} \tag{85}$$

Also, there will be another boundary condition

$$\int_1^{1+\bar{l}} e_x \, dx = \int_1^{1+\bar{l}} \bar{u}_x \, dx = 0 \tag{86}$$

From equation (86) the value of G is found to be

$$G = \frac{1}{\log(1+\bar{l})} \left\{ \int_1^{1+\bar{l}} \left[\frac{b_{12}}{b_{11}} \cot\alpha \, \frac{\bar{\omega}}{\bar{x}} - \frac{c_1 \theta}{b_{11}} \right] dx \right\} \tag{87}$$

and the axial stress will be given by

$$\sigma_x = \frac{G \, b_{11}}{\bar{x}} \tag{88}$$

SECTION VI

THERMAL BUCKLING OF SHELLS

VI.1 Introduction

The theory of buckling deals mainly with the conditions under which the equilibrium ceases to be stable. A class Γ of unbuckled configuration, corresponding to a range of values of a real parameter p is first considered. This p may be a mechanical load or a thermal load. For each value of p in the range of interest there corresponds a single configuration in class Γ. In the classical problem of buckling, the configuration in class Γ is stable when p is small. But when p is increased to a certain critical value, configuration in class Γ ceases to be stable. The problem is to determine the critical value of p.

VI.1.1 Energy Theory of Buckling

The energy theory of buckling propounded by Bryan (20) was based on the law that a static conservative system is in a state of stable equilibrium, if and only if, the value of potential energy is at a relative minimum. According to the principle of virtual work, for any equilibrium condition, the first variation of potential energy δV vanishes. If in addition, the second variation of potential energy is positive definite, the equilibrium

is stable(1). At the buckling load, $\delta^2 V$ ordinarily changes its character from positive definiteness to negative definiteness, negative semi-definiteness or indefiniteness. Thus it may be anticipated that $\delta^2 V$ is positive semi-definite when the buckling load is reached. From there it follows, that during buckling, there exists non-zero virtual displacements for which $\delta^2 V = 0$

The total potential energy of a loaded structural shell is the sum of strain energy and potential energy due to external load. Since we are only concerned with thermal buckling, we will consider strain energy expression as total potential energy expression. In the following paragraph, an expression for second variation of strain energy which is applicable for buckling analysis of shells will be developed. In the development of second variation expression, non-linear stretching of the middle surface will be considered. Since it will be impractical to consider all the non-linear terms, certain assumptions will be made to simplify the second variation expression.

The strain energy expression for an orthotropic shell is given by equation (29), Section IV,

$$U = U_m + U_b + U_\theta$$

where,

$$U_m = \frac{1}{2} \iint \left[b_{11} e_x^2 + b_{22} e_y^2 + b_{66} e_{xy}^2 + 2 b_{12} e_x e_y \right] h A B\, dx\, dy \tag{1}$$

$$U_b = \frac{1}{24} \iint \left[b_{11} K_x^2 + b_{66} K_{xy}^2 + b_{22} K_y^2 + 2 b_{12} K_x K_y \right] h^3 A B\, dx\, dy \tag{2}$$

$$U_\theta = - \iint \left[(C_1 e_x + C_2 e_y) \theta_0 + (C_1 K_x + C_2 K_y) \theta_1 \right] A B\, dx\, dy \tag{3}$$

where,

$$\theta_0 = \int_{-h/2}^{h/2} \theta\, dz \qquad\qquad \theta_1 = \int_{-h/2}^{h/2} \theta z\, dz \tag{4}$$

A look at equation (26) of Section IV reveals that the middle surface strains can be written in the following form:

$$e_x = \bar{e}_x + \frac{\omega_x^2}{2 A^2}$$

$$e_y = \bar{e}_y + \frac{\omega_y^2}{2 B^2}$$

$$e_{xy} = \bar{e}_{xy} + \frac{\omega_x \omega_y}{A B} \tag{5}$$

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where $\bar{e}_x$, $\bar{e}_y$ and $\bar{e}_{xy}$ are linear functions of displacement components u,v, w and their derivatives. The curvature K_x, K_y and K_{xy} are linear functions of the derivatives of w.

In order to determine the second variation of strain energy, virtual increments are given to displacement components. The terms, which will be quadratic in these virtual displacements, will constitute the second variation.

Thus the change in potential energy can be written as

$$\triangle U = \delta U + \frac{1}{2!} \delta^2 U + \cdots \tag{6}$$

The second variation,

$$\delta^2 U = \iint \left\{ b_{11} \left(\delta \bar{e}_x\right)^2 + b_{22}\left(\delta \bar{e}_y\right)^2 + b_{66}\left(\delta \bar{e}_{xy}\right)^2 + 2 b_{12}\left(\delta \bar{e}_x\right) \right.$$

$$\left(\delta \bar{e}_y\right) + \left(b_{11}\bar{e}_x + b_{12}\bar{e}_y\right)\frac{\left(\delta\omega_x\right)^2}{A^2} + \left(b_{12}\bar{e}_x + b_{22}\bar{e}_y\right)\frac{\left(\delta\omega_y\right)^2}{2B^2}$$

$$+ b_{66}\,\bar{e}_{xy}\frac{\delta\omega_x\,\delta\omega_y}{A\,B} + 2\left[\left(b_{11}\delta\bar{e}_x + b_{12}\delta\bar{e}_y\right)\left(\frac{\delta\omega_x}{A^2}\right)\omega_x + \right.$$

$$\left. \left(b_{12}\delta\bar{e}_x + b_{22}\delta\bar{e}_y\right)\left(\frac{\delta\omega_y}{B^2}\right)\omega_y + b_{66}\,\delta e_{xy}\left(\frac{\omega_x\,\delta\omega_y}{A\,B}\right.\right.$$

$$\left.\left.\left. + \frac{\omega_y\,\delta\omega_x}{A\,B}\right)\right]\right\} h\,A\,B\,dx\,dy$$

$$+ \frac{1}{12}\iint \left[b_{22}\left(\delta K_y\right)^2 + b_{66}\left(\delta K_{xy}\right)^2 + b_{11}\left(\delta K_x\right)^2 \right.$$

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$$+ 2b_{12}(\delta K_x)(\delta K_y)] h^3 AB\, dx\, dy$$

$$-\iint \left[C_1 \theta_0 \frac{(\delta \omega_x)^2}{A^2} + C_2 \theta_0 \frac{(\delta \omega_y)^2}{B^2} \right] AB\, dx\, dy \qquad (7)$$

The above expressions are obtained from the consideration that the prebuckling displacements obey the linear elasticity law. Thus the squares and products of prebuckling rotations have been neglected, since they are very small compared to prebuckling elongation and shear.

Considering the second variation of membrane energy, the terms given in the bracket [ ], which contains prebuckling rotations, will vanish if there is no bending present in the pre-buckling configuration. In general, for long cylinders, the individual terms in the bracket [ ] are small if compared to other terms in the second variation expression, but may not be small enough that they can be neglected. However their net effect on the entire surface of the shell is small.

Buckling Analysis

For buckling analysis e_x, e_y and e_{xy} are to be expressed in terms of u,v, w and their derivatives. $\delta \bar{e}_x, \delta \bar{e}_y, \delta \omega_x, \delta K_x$ etc. can be expressed in terms of u',v', w' and their derivatives, where u',v' and w' are buckling displacements. Now certain

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buckling displacement functions are to be assumed. For simple supported case, the displacement function can be approximated by,

$$u' = \cos my \sum_{n=1}^{N} U_n \cos \frac{n\pi x}{l}$$

$$v' = \sin my \sum_{n=1}^{N} V_n \sin \frac{n\pi x}{l}$$

$$w' = \cos my \sum_{n=1}^{N} W_n \sin \frac{n\pi x}{l} \qquad (8)$$

The vanishing of the first variation of total potential energy yields three equilibrium equations. From these three equilibrium equations, after certain manipulations, U_n and V_n can be expressed in terms of W_n. At this stage u', v' and w' are substituted in the second variation expression.

Also u, v, w etc., as calculated in terms of Θ from prebuckling equilibrium equation, are substituted in the expression for $\delta^2 U$.

Thus after integration we obtain,

$$\delta^2 U = \sum_{p=1}^{N} \sum_{n=1}^{N} A_{pn} W_p W_n \qquad (9)$$

where, A_{pn} contains Θ terms.

The condition for the limit of positive definiteness of the quadratic form is that the determinant of the co-efficients be equal to zero. Therefore, the buckling problem reduces to an eigen value problem. The value of N depends upon the accuracy desired and is to be determined by trial. For most cases results converge very rapidly.

There may be another way of looking at the buckling problem from energy considerations. One considers first the change in potential energy ΔU from a peculiar state of equilibrium which exists at the incipience of buckling. For equilibrium, the first variation of potential energy must be zero. If the first variation of ΔU is equated to zero, then a condition for which buckling configuration is an equilibrium configuration is obtained. This condition in turn indicates that the prebuckling configuration is a neutral equilibrium. The condition for buckling is obtained in the following way. The change in potential energy is expressed in terms of U_n, V_n and W_n by substituting the assumed form of buckling displacements given by equation (8). The integration is carried out to arrive at an algebraic expression involving unknown coefficients U_n, V_n and W_n. The requirement of zero variation of the integration is hence replaced by minimization with respect to U_n, V_n and

W_n of the algebraic expression. This leads to a set of simultaneous equations the number of which depends upon N. Non-trivial solutions for U_n, V_n and W_n exist, if the determinant of the co-efficients in the above mentioned equations is zero. N can be increased and hence the order of determinant, until the satisfactory convergence is achieved.

VI.2   Thermal Buckling of Cylindrical Shell

Considerable work has been done in the field of thermal buckling of isotropic shells. Hoff[21] and Zuk[22] considered the case of uniform temperature rise and solved the buckling problem of a cylindrical shell which is restrained circumferentially but free from constraints in axial direction. The case of cylindrical shells subjected to axial temperature distribution has been solved by Sunakawa[23]. The buckling of cylindrical shells heated along an axial strip has been studied in a series of reports (24-25).

In this study instability of orthotropic shells subjected to a temperature distribution will be considered. Investigation will be made for two types of boundary conditions - simple supported and fixed end.

Temperature distributions for which buckling criteria will be investigated are assumed to be axi-symmetric. The temperature will be assumed to vary axially or radially.

The buckling criteria will be determined by enforcing the condition that the change of potential energy during buckling is minimum. Neglecting the higher order terms, the change in potential energy during buckling can be written as,

$$\Delta U = \int_0^{2\pi} \int_0^{\ell} \left\{ b_{11} (u_x)^2 + b_{22} \left(\frac{v_y + \omega}{a}\right)^2 + b_{66} \left(v_x + \frac{u_y}{a}\right)^2 \right.$$

$$\left. + 2b_{12} (u_x)\left(\frac{v_y + \omega}{a}\right) \right\} ha\,dx\,dy$$

$$+ \frac{1}{2} \int_0^{2\pi} \int_0^{\ell} \left[N_x (\omega_x)^2 + N_y \left(\frac{\omega_y}{a}\right)^2 \right] a\,dx\,dy$$

$$+ \int_0^{2\pi} \int_0^{\ell} \left\{ \left[b_{11} u_x + b_{12} \left(\frac{v_y + \omega}{a}\right) \right] \omega_x \bar{\omega}_x \right.$$

$$\left. + b_{66} \left[v_x + \frac{u_y}{a} \right] \left(\frac{\bar{\omega}_x \omega_y}{a} \right) \right\} ha\,dx\,dy$$

$$+ \frac{1}{24} \int_0^{2\pi} \int_0^{\ell} \left[b_{11} \omega_{xx}^2 + \frac{b_{22}}{a^4} \omega_{yy}^2 + \frac{4 b_{66}}{a^2} \omega_{xy}^2 \right.$$

$$\left. + \frac{2 b_{12} \omega_{xx} \omega_{yy}}{a^2} \right] h^3 a\,dx\,dy \qquad (10)$$

where,

$$N_x = (b_{11} \bar{e}_x + b_{12} \bar{e}_y) h - C_1 \theta_0$$

$$N_y = (b_{12} \bar{e}_x + b_{22} \bar{e}_y) h - C_2 \theta_0 \qquad (11)$$

e_x and e_y indicate middle surface strain. The bar over the quantities indicates quantities prior to buckling. We note that expression ΔU is quadratic in terms of displacements. The absence of the first order terms lie in the fact that pre-buckling configuration is an equilibrium configuration. We also note ΔU contains pre-buckling rotations.

(A) Simple Supported Case

At this stage we assume the following functions for

$$U = \cos my \sum_{n=1}^{N} U_n \cos \frac{n\pi x}{l}$$

$$V = \sin my \sum_{n=1}^{N} V_n \sin \frac{n\pi x}{l}$$

$$\omega = \cos my \sum_{n=1}^{N} W_n \sin \frac{n\pi x}{l}$$

where, m is an integer. Such a solution describes a buckling mode with m circumferential lobes.

Substituting the displacement functions in change of potential energy ΔU and integrating, the following algebraic expression which involves U_n, V_n and W_n is obtained.

$$\Delta U = \sum_{n'=1}^{N} \sum_{n=1}^{N} \left[\frac{b_{22}}{2a^2} V_n V_{n'} m^2 + \frac{b_{11}}{2l^2} nn' \pi^2 U_n U_{n'} \right.$$

$$+ \frac{b_{22}}{2a^2} W_n W_{n'} + \frac{b_{22}}{a^2} m V_n W_{n'} + b_{66} \frac{nn'\pi^2}{2l^2} V_n V_{n'}$$

$$+ \frac{b_{66}}{2a^2} m^2 U_n U_{n'} - \frac{b_{66}}{al} mn\pi V_n U_{n'} - \frac{b_{12}}{al} mn\pi U_n V_{n'}$$

$$- \frac{b_{12}}{al} n\pi U_n W_{n'} \Big] \frac{h\pi la}{2} \delta_n^{n'} + \Big[b_{11} W_n W_{n'} \frac{n'^2 n^2 \pi^4}{2l^4} +$$

$$\frac{4b_{66}}{2a^2 l^2} m^2 nn'\pi^2 W_n W_{n'} + \frac{b_{22} m^4}{2a^4} W_n W_{n'} + \frac{b_{12}}{a^2 l^2} n^2 m^2 \pi^2 W_n W_{n'} \Big]$$

$$\times \frac{l\pi h^3 a}{24} \delta_n^{n'} + R_2 \frac{m^2 h\pi}{4a} H_1{}_{n'}^n W_n W_{n'} + \frac{nn' b_{11}\pi^3}{2l^2} ha K H_2{}_{n'}^n \cdot$$

$$U_n W_{n'} + \frac{ha}{2al} b_{12} m\pi^2 K n' H_2{}_{n'}^n V_n W_{n'} + \frac{ha}{2al} n' b_{12} \cdot$$

$$\pi^2 K H_2{}_{n'}^n W_n W_{n'} + b_{66} \frac{\pi m^2 K}{2a^2} H_2{}_{n'}^n ha W_{n'} U_n - \frac{m b_{66}}{2al} \cdot$$

$$\pi^2 n V_n W_{n'} K H_2{}_{n'}^n ha . \Big\}$$

(13)

where,

$$R_2 = \left(\frac{b_{22}}{a} - \frac{b_{12}^2}{a b_{11}} \right)$$

(14)

$$H_{1,n'}^{n} = -\sum_{i=1}^{2}(-1)^{i}\left\{ B_{1}\,\bar{e}^{k\ell}\left\{\dfrac{[K-\Delta i]\sin[K-\Delta i]\ell - K\cos[K-\Delta i]\ell}{2\{K^{2}+(K-\Delta i)^{2}\}}\right.\right.$$

$$+ B_{1}\,\bar{e}^{k\ell}\left\{\dfrac{[K+\Delta i]\sin[K+\Delta i]\ell - K\cos[K+\Delta i]\ell}{2\{K^{2}+(K+\Delta i)^{2}\}}\right\}$$

$$+ B_{2}\,\bar{e}^{k\ell}\left\{\dfrac{-K\sin[K-\Delta i]\ell - [K-\Delta i]\cos[K-\Delta i]\ell}{2\{K^{2}+(K-\Delta i)^{2}\}}\right\}$$

$$+ B_{2}\,\bar{e}^{k\ell}\left\{\dfrac{-K\sin[K+\Delta i]\ell - [K+\Delta i]\cos[K+\Delta i]\ell}{2\{K^{2}+(K+\Delta i)^{2}\}}\right\}$$

$$+ B_{3}\,e^{k\ell}\left\{\dfrac{[K-\Delta i]\sin[K-\Delta i]\ell + K\cos[K-\Delta i]\ell}{2\{K^{2}+[K-\Delta i]^{2}\}}\right\}$$

$$+ B_{3}\,e^{k\ell}\left\{\dfrac{[K+\Delta i]\sin[K+\Delta i]\ell + K\cos[K+\Delta i]\ell}{2\{K^{2}+(K+\Delta i)^{2}\}}\right\}$$

$$+ B_{4}\,e^{k\ell}\left\{\dfrac{K\sin[K-\Delta i]\ell - [K-\Delta i]\cos[K-\Delta i]\ell}{2\{K^{2}+(K-\Delta i)^{2}\}}\right\}$$

$$+ B_{4}\,e^{k\ell}\left\{\dfrac{K\sin[K+\Delta i]\ell - [K+\Delta i]\cos[K+\Delta i]\ell}{2\{K^{2}+(K+\Delta i)^{2}\}}\right\}$$

$$+ \frac{B_1 K}{2\{K^2 + (K-\Delta i)^2\}} + \frac{B_1 K}{2\{K^2 + (K+\Delta i)^2\}} + \frac{B_2 (K-\Delta i)}{2\{K^2 + (K-\Delta i)^2\}}$$

$$+ \frac{B_2 (K+\Delta i)}{2\{K^2 + (K+\Delta i)^2\}} - \frac{B_3 K}{2\{K^2 + (K-\Delta i)^2\}} - \frac{B_3 K}{2\{K^2 + (K+\Delta i)^2\}}$$

$$\left. + \frac{B_4 (K-\Delta i)}{2\{K^2 + (K-\Delta i)^2\}} + \frac{B_4 (K+\Delta i)}{2\{K^2 + (K+\Delta i)\}} \right\} \tag{15}$$

$$\Delta_1 = (n'-n)\pi/\ell \tag{16}$$

$$\Delta_2 = (n'+n)\pi/\ell \tag{17}$$

$$\delta_n^{n'} = 1 \qquad \text{For} \quad n' = n \tag{18}$$

$$\delta_n^{n'} = 0 \qquad \text{For} \quad n' \neq n \tag{19}$$

$$H_{2\,n'}^{\,n} = \sum_{i=1}^{2} \left\{ -(B_1 + B_2) \left\{ e^{-K\ell} \frac{[(K-\Delta i)\sin(K-\Delta i)\ell - K\cos(K-\Delta i)\ell]}{2[K^2 + (K-\Delta i)^2]} \right.\right.$$

$$\left.\left. - e^{-K\ell} \frac{[(K+\Delta i)\sin(K+\Delta i)\ell - K\cos(K+\Delta i)\ell]}{2[K^2 + (K+\Delta i)^2]} \right\} \right.$$

$$+ (B_2 - B_1) \left\{ e^{-K\ell} \frac{[-K\sin(\Delta i - K)\ell - (\Delta i - K)\cos(\Delta i - K)\ell]}{2[K^2 + (\Delta i - K)^2]} \right.$$

$$+ e^{-K\ell} \frac{[-K\sin(\Delta i+K)\ell - (\Delta i+K)\cos(\Delta i+K)\ell]}{2[K^2 + (\Delta i+K)^2]} \Bigg\}$$

$$+ (B_3+B_4) \Bigg\{ e^{K\ell} \frac{[K\sin(\Delta i-K)\ell - (\Delta i-K)\cos(\Delta i-K)\ell]}{2[K^2 + (\Delta i-K)^2]}$$

$$+ e^{K\ell} \frac{[K\sin(\Delta i+K)\ell - (\Delta i+K)\cos(\Delta i+K)\ell]}{2[K^2 + (\Delta i+K)^2]} \Bigg\}$$

$$+ (B_4-B_3) \Bigg\{ e^{K\ell} \frac{[(K-\Delta i)\sin(K-\Delta i)\ell + K\cos(K-\Delta i)\ell]}{2[K^2 + (K-\Delta i)^2]}$$

$$- e^{K\ell} \frac{[(K+\Delta i)\sin(K+\Delta i)\ell + K\cos(K+\Delta i)\ell]}{2[K^2 + (K+\Delta i)^2]} \Bigg\}$$

$$+ (B_1+B_2) \Bigg\{ \frac{-K}{2[K^2 + (K-\Delta i)^2]} + \frac{K}{2[K^2 + (K+\Delta i)^2]} \Bigg\}$$

$$+ (B_2-B_1) \Bigg\{ \frac{\Delta i-K}{2[K^2 + (\Delta i-K)^2]} + \frac{\Delta i+K}{2[K^2 + (\Delta i+K)^2]} \Bigg\}$$

$$+ (B_3+B_4) \Bigg\{ \frac{\Delta i-K}{2[K^2 + (\Delta i-K)^2]} + \frac{\Delta i+K}{2[K^2 + (\Delta i+K)^2]} \Bigg\}$$

$$- (B_4-B_3) \Bigg\{ \frac{K}{2[K^2 + (K-\Delta i)^2]} - \frac{K}{2[K^2 + (K+\Delta i)^2]} \Bigg\} \Bigg\}$$

$$(20)$$

Minimizing the change in the potential energy expression given by equation (13) with respect to U_p, V_p and W_p; i.e by setting $\dfrac{\partial \Delta U}{\partial U_p} = \dfrac{\partial \Delta U}{\partial V_p} = \dfrac{\partial \Delta U}{\partial W_p} = 0$,

$$\sum_{n=1,2,}^{N} \left(A'_{n1} U_n + B'_{n1} V_n + C'_{n1} W_n \right) = 0$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$\sum_{n=1,2}^{N} \left(A'_{nN} U_n + B'_{nN} V_n + C'_{nN} W_n \right) = 0$$

$$\sum_{n=1,2}^{N} \left(A^2_{n1} U_n + B^2_{n1} V_n + C^2_{n1} W_n \right) = 0$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$\sum_{n=1,2}^{N} \left(A^2_{nN} U_n + B^2_{nN} V_n + C^2_{nN} W_n \right) = 0$$

$$\sum_{n=1,2}^{N} \left(A^3_{n1} U_n + B^3_{n1} V_n + C^3_{n1} W_n \right) = 0$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$\sum_{n=1,2}^{N} \left(A^3_{nN} U_n + B^3_{nN} V_n + C^3_{nN} W_n \right) = 0$$

$$(21)$$

where,

$$A'_{np} = \frac{b_{11}\pi^2 np}{\ell^2}\,\delta_n^p + \frac{b_{66}m^2}{a^2}\,\delta_n^p \tag{22}$$

$$B'_{np} = -\left(\frac{b_{66}mn\pi}{a\ell} + \frac{b_{12}mn\pi}{a\ell}\right)\delta_n^p \tag{23}$$

$$C'_{np} = -\frac{b_{12}n\pi}{a\ell}\,\delta_n^p - \frac{b_{11}np\pi^2 K}{\ell^3}\,H_2^p{}_n + \frac{b_{66}m^2 K}{a^2\ell}\,H_2^p{}_n \tag{24}$$

$$A^2_{np} = B'_{pn} \tag{25}$$

$$B^2_{np} = \frac{b_{22}m^2}{a^2}\,\delta_p^n + \frac{b_{66}np\pi^2}{\ell^2}\,\delta_p^n \tag{26}$$

$$C^2_{np} = \frac{b_{22}m}{a^2}\,\delta_p^n + \frac{b_{12}nm\pi K}{a\ell^2}\,H_2^p{}_n - \frac{b_{66}mp\pi K}{a\ell^2}\,H_2^p{}_n \tag{27}$$

$$A^3_{np} = C'_{pn} \tag{28}$$

$$B^3_{np} = C^2_{pn} \tag{29}$$

$$C^3_{np} = \frac{b_{22}}{a^2}\,\delta_p^n + \left(\frac{b_{11}p^2n^2\pi^4}{\ell^4} + \frac{4\,b_{66}m^2np\pi^2}{a^2\ell^2}\right.$$
$$\left. + \frac{b_{22}m^4}{a^4} + \frac{b_{12}n^2m^2\pi^2}{a^2\ell^2} + \frac{b_{12}p^2m^2\pi^2}{a^2\ell^2}\right)\frac{h^2}{12}\,\delta_p^n$$
$$+ \frac{n b_{12}\pi K}{a\ell^2}\,H_2^n{}_p + \frac{p b_{12}\pi K}{a\ell^2}\,H_2^p{}_n + Y \tag{30}$$

Y = Co-efficient of W_n in $\dfrac{\partial}{\partial W_p}\left(R_2 W_n W_{n'} m^2 h\pi H^n_{n'}\right)/2h\pi la^2$. (31)

It may be noted that the term $H^n_{2\ p}$ takes care of pre-buckling rotation. Setting $H^n_{2\ p}=0$, a buckling criteria, that does not consider pre-buckling rotation, can be obtained.

Neglecting the effect of pre-buckling rotation, when N=2, the buckling temperature can be obtained from the following equation

$$B\theta^2 + C\theta + K = 0 \qquad (32)$$

The constants B, C and K will be given by the following equations

$$B = \left(G^3_{21}G^3_{12} - G^3_{22}G^3_{11}\right)\Delta^{33}_{12} \qquad (33)$$

$$C = G^3_{21}\left(C^2_{11}\Delta^{32}_{11} + C^1_{11}\Delta^{31}_{11} - C^2_{12}\Delta^{32}_{12} + D^3_{12}\Delta^{33}_{12}\right)$$

$$+ G^3_{22}\left(C^2_{12}\Delta^{32}_{22} - C^1_{11}\Delta^{31}_{21} - C^2_{11}\Delta^{32}_{21} - D^3_{11}\Delta^{33}_{21}\right)$$

$$+ G^3_{11}\left(C^2_{22}\Delta^{32}_{12} + C^1_{22}\Delta^{31}_{12} - C^2_{21}\Delta^{32}_{11} - D^3_{22}\Delta^{33}_{12}\right)$$

$$+ G^3_{12}\left(C^2_{21}\Delta^{32}_{21} - C^1_{22}\Delta^{31}_{22} - C^2_{22}\Delta^{32}_{22} + D^3_{21}\Delta^{33}_{21}\right) \qquad (34)$$

$$K = C^1_{22}\left(C^2_{11}\Delta^{12}_{21} - C^1_{11}\Delta^{11}_{21} - C^2_{12}\Delta^{12}_{22} + D^3_{11}\Delta^{13}_{21} - D^3_{12}\Delta^{23}_{12}\right)$$

$$+ \ C_{21}^2 \left(C_{11}^1 \, \Delta_{11}^{21} + C_{12}^2 \, \Delta_{21}^{22} - D_{11}^3 \, \Delta_{11}^{23} + D_{12}^3 \, \Delta_{12}^{23} \right)$$

$$+ \ C_{22}^2 \left(- C_{11}^1 \, \Delta_{21}^{21} - C_{11}^2 \, \Delta_{21}^{22} + D_{11}^3 \, \Delta_{21}^{23} - D_{12}^3 \, \Delta_{22}^{23} \right)$$

$$+ \ D_{21}^3 \left(C_{11}^1 \, \Delta_{11}^{31} + C_{11}^2 \, \Delta_{11}^{32} - C_{12}^2 \, \Delta_{12}^{32} + D_{12}^3 \, \Delta_{12}^{33} \right)$$

$$+ \ D_{22}^3 \left(- C_{11}^1 \, \Delta_{21}^{31} - C_{11}^2 \, \Delta_{21}^{32} + C_{12}^2 \, \Delta_{22}^{32} - D_{11}^3 \, \Delta_{21}^{33} \right) \tag{35}$$

where,

$$G_{11}^3 = \left(Y_{\substack{n=1 \\ p=1}} \right) \Big/ \theta \tag{36}$$

$$G_{21}^3 = \left(Y_{\substack{n=2 \\ p=1}} \right) \Big/ \theta \tag{37}$$

$$G_{12}^3 = \left(Y_{\substack{n=1 \\ p=2}} \right) \Big/ \theta \tag{38}$$

$$G_{22}^3 = \left(Y_{\substack{n=2 \\ P=2}} \right) \Big/ \theta \tag{39}$$

Δ_{nm}^{rp} is a fourth order determinant obtained from basic determinant (Equation 22) by eliminating the last two columns and eliminating rows containing elements R_{in}^r and R_{im}^p.

$$D_{np}^3 = C_{np}^3 - Y - \frac{n b_{12} \pi K}{a \ell^2} H_{2 p}^n - \frac{p b_{12} \pi K}{a \ell^2} H_{n 2}^p \tag{40}$$

If the effect of pre-buckling rotation is neglected, then because of the presence of Kronecker delta in the co-efficients of U_n, V_n etc., it is possible to express the buckling criteria in the following form:

$$
\begin{vmatrix}
tR_1 + S_1^1 & S_2^1 & S_3^1 & \cdots \\
S_1^2 & tR_2 + S_2^2 & S_3^2 & \cdots \\
S_1^3 & S_3^2 & tR_3 + S_3^3 & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{vmatrix} = 0 \qquad (41)
$$

where,

$$
R_n = \left(- A_{nn}^2 \frac{b_{12} b_{22} mn\pi}{a^3 \ell} - B_{nn}^2 \frac{b_{12}^2 n^2 \pi^2}{a^2 \ell^2} - B_{nn}^1 \cdot \right.
$$

$$
\left. \frac{b_{12} b_{22} mn\pi}{a^3 \ell} - A_{nn}^1 \frac{b_{22}^2 m^2}{a^4} \right) \Big/ \left(A_{nn}^1 B_{nn}^2 - A_{nn}^2 B_{nn}^1 \right) + D_{nn}^3
$$

$$
\theta = 1/t \qquad (43)
$$

$$
S_{n'}^n = H_{n'}^n \frac{R_2 m^2}{a^2 \ell \theta}
$$

When the temperature distribution is symmetrical about the mid-plane of the shell, the buckling occurs either in a symmetric or an anti-symmetric pattern. The condition for symmetric buckling can be obtained from the following determinant:

$$
\begin{vmatrix}
tR_1 + S_1^1 & S_3^1 & S_5^1 & \cdots & S_n^1 \\
S_1^3 & tR_3 + S_3^3 & S_5^3 & \cdots & S_n^{n-2} \\
S_1^5 & S_5^3 & tR_5 + S_5^5 & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \\
S_1^n & S_n^{n-2} & \cdots & & tR_n + S_n^n
\end{vmatrix} = 0 \qquad (46)
$$

The value n in the above determinant is to be selected by trial until the desired accuracy in the result is obtained.

(B) Fixed End Case

For fixed end condition the following displacement pattern can be assumed.

$$
u = \sin my \sum_{n=1}^{N} U_n \sin \frac{2n\pi x}{\ell}
$$

$$
v = \cos my \sum_{n=1}^{N} V_n \left(1 - \cos \frac{2n\pi x}{\ell} \right)
$$

$$\omega = \text{Sin } my \sum_{n=1}^{N} W_n \left(1 - \cos \frac{2n\pi x}{\ell}\right) \tag{47}$$

Such displacement functions satisfy the forced boundary conditions for fixed end. These conditions are:

$$u = 0 \quad , \quad \upsilon = 0 \; , \quad \omega = 0 \qquad @ \; x = 0 \; \text{ and } x = \ell$$

and

$$\omega_x = 0 \qquad @ \; x = 0 \text{ and } x = \ell$$

Substituting the displacement function in change of potential energy, $\triangle U$, during buckling and integrating, we obtain,

$$\triangle U = \sum_{n=1}^{N} \sum_{n'=1}^{N} \left\{ \frac{b_{22}}{2a^2} m^2 V_n V_{n'} + 2nn'\pi^2 \frac{b_{11}}{\ell^2} U_n U_{n'} \right.$$

$$+ \frac{b_{22}}{2a^2} W_n W_{n'} - 2 \frac{b_{22}}{2a^2} m V_n V_{n'} + 2 nn'\pi^2 \frac{b_{66}}{\ell^2} V_n V_{n'}$$

$$+ \frac{b_{66}}{2a^2} m^2 U_n U_{n'} + 2 \frac{b_{66}}{a\ell} mn'\pi V_{n'} U_n + 2n\pi m \frac{b_{12}}{a\ell} U_n V_{n'}$$

$$\left. - 2 \frac{b_{12}}{a\ell} n\pi W_{n'} U_n \right\} \frac{h\pi \ell a}{2} \delta_{n'}^{n} + \left\{ 8 \frac{b_{11}}{\ell^4} n^2 n^2 \pi^4 W_n W_{n'} \right.$$

$$+ 8 \frac{b_{66}}{a^2 \ell^2} m^2 n' n \pi^2 W_n W_{n'} + \frac{b_{22}}{2a^4} m^4 W_n W_{n'} + 4 \frac{b_{12}}{a^2 \ell^2} n^2.$$

$$\pi^2 m^2 W_n W_{n'} \,\Big]\, \frac{h^3 \pi \ell a}{24} \delta^n_{n'} \;+\; \Big[\, \frac{b_{22}}{a^2} W_n W_{n'} \;+\; \frac{b_{22}}{a^2} m^2.$$

$$. V_n V_{n'} \;-\; \frac{b_{22}}{a^2} m\, V_n W_{n'} \Big]\, \frac{h \pi \ell a}{2} \;+\; \frac{b_{22}}{a^4} m^4 \frac{h^3}{24} \pi \ell a .$$

$$W_n W_{n'} \;+\; 2 \frac{b_{11}}{\ell^2} K n n' \pi^3 h a\, H_2^{\,n'}{}_{n'} U_n W_{n'} \;-\; \frac{b_{12}}{a\ell} m h a n' \pi^2 V_n W_{n'}$$

$$K\, H_2^{\,n''}{}_{n'} \;+\; \frac{b_{12}}{a\ell} h a n' \pi^2 K\, H_2^{\,n''}{}_{n'} W_n W_{n'} \;+\; \frac{b_{66}}{a\ell} h a \pi m n \pi^2 K\, H_2^{\,n''}{}_{n'} V_n W_{n'}$$

$$+\; \frac{b_{66}}{2a^2} h a m^2 \pi K\, H_2^{\,n''}{}_{n'} \;+\; R_2 \frac{m^2 h \pi a}{2 a^2} H_3^{\,n}{}_{n'} W_n W_{n'} \;+\; 4 \frac{n n'}{2\ell^3} \pi^3.$$

$$\sigma_x . h a\, \ell\, W_n W_{n'}. \tag{48}$$

where, $H_2^{\,n'}{}_{n'} = \displaystyle\sum_{i=1}^{2} \Big[\text{expression in the bracket} \{\ \} \text{of equation (20)} \Big]$ (49)

$$\Delta_1 = 2(n-n')\,\pi/\ell \qquad\qquad \Delta_2 = 2(n+n')\,\pi/\ell \tag{50}$$

$$H_2^{\,n''}{}_{n'} = \displaystyle\sum_{i=1}^{3} (-1)^i \Big[\text{expression in the bracket} \{\ \} \text{of equation (20)} \Big] \tag{51}$$

$$\Delta_1 = 2(n-n')\,\pi/\ell \qquad \Delta_2 = 2n\pi/\ell \qquad \Delta_3 = 2(n+n')\,\pi/\ell \tag{52}$$

$$H_3^{\,n}{}_{n'} = \displaystyle\sum_{i=1}^{3} (-1)^i\, H_3' \;+\; \frac{1}{2} \displaystyle\sum_{i=4}^{5} H_3' \tag{53}$$

$H_3^!$ is the expression in the bracket$\{\ \}$ of equation (15)

$$\Delta_1 = 2n\pi/\ell \qquad \Delta_2 = 0 \qquad \Delta_3 = 2n'\pi/\ell$$

$$\Delta_4 = 2(n'-n)\pi/\ell \qquad \Delta_5 = 2(n+n')\pi/\ell \tag{54}$$

Minimizing the change in the potential energy expression given by equation (48) with respect to U_p, V_p and W_p, an equation like (21) is obtained, where,

$$A'_{np} = \frac{4 b_{11} n p \pi^2}{\ell^2} \delta_n^p + \frac{b_{66} m^2}{a^2} \delta_n^p$$

$$B'_{np} = \left(2 \frac{b_{66} m n \pi}{a\ell} + 2 \frac{b_{12} m n \pi}{a\ell} \right) \delta_n^p$$

$$C'_{np} = -2 \frac{b_{12} n \pi}{a\ell} \delta_n^p + \frac{4 K b_{11} n p \pi^2}{\ell^3} H'^p_{n2} + b_{66} \frac{m^2 K}{a^2\ell} H''^p_{n2}$$

$$A^2_{pn} = B'_{np}$$

$$B^2_{np} = \left(\frac{b_{22} m^2}{a^2} + \frac{4 n p \pi^2 b_{66}}{\ell^2} \right) \delta_n^p + \frac{2 b_{22} m^2}{a^2}$$

$$C^2_{np} = -\frac{b_{22} m}{a^2} \delta_n^p - \frac{2 b_{22} m}{a^2} - \frac{2 m n \pi K b_{12}}{a\ell^2} H''^p_{n2}$$

$$+ \frac{2b_{66}\, K_m \pi}{a\ell^2} \overset{p}{\underset{n}{H}}{}''_2$$

$$A^3_{np} = C^1_{pn}$$

$$B^3_{np} = C^2_{pn}$$

$$C^3_{np} = \frac{3b_{22}}{a^2} + \left(\frac{16\, b_{11}\, p^2 n^2 \pi^4}{\ell^4} + \frac{16\, b_{66}\, m^2 n p \pi^2}{a^2 \ell^2} + \frac{m^4 b_{22}}{a^4} \right.$$

$$\left. + \frac{4\, n^2 \pi^2 m^2\, b_{12}}{a^2 \ell^2} + \frac{4\, p^2 m^2 \pi^2 b_{12}}{a^2 \ell^2} \right) \frac{h^2}{12} \delta^n_p + Y$$

$$+ \frac{2\, m^4 b_{22}}{a^4}\frac{h^2}{12} + \frac{2p\, b_{12}\, K}{a\ell^2} \overset{n}{\underset{p}{H}}{}''_2 + \frac{2n\, b_{12}\, K}{a\ell^2} \overset{p}{\underset{n}{H}}{}''_2 .$$

$$Y = \text{Co-efficient of } W_n \text{ in } \frac{\partial}{\partial W_p} \sum \sum W_n W_{n'} \left[\frac{R_2\, m^2 \overset{n}{H}{}^3_{n'}}{2a^2 \ell} + \frac{4nn' \pi^2 \sigma_x}{\ell^2} \right]$$

VI.3 Thermal Buckling of Conical Shells

The problem of buckling of conical shells has been examined by many investigators. Thermal buckling problem of isotropic conical shells has been studied by Singer and Hoff(19) and Lu and Change(16). However to this investigator's knowledge no significant work has been done in the case of thermal buckling of orthotropic conical shells.

As it has been done before, the critical temperature distribution will be determined by enforcing the condition that the change in potential energy during buckling is minimum. The displacement functions that will be used in this case are the same as those used in the case of cylindrical shells. These displacement functions were first used by Niordson(27) for conical shells and are found to be quite satisfactory for cases where the slant length of the cone is small in comparison to base diameter.

Theoretical Analysis

For simple supported case, displacement functions may be assumed in the following form

$$\bar{u} = \cos my \sum_{n=1}^{N} U_n \cos n\pi (\bar{x} - 1)/\bar{l}$$

$$\bar{v} = \sin my \sum_{n=1}^{N} V_n \sin n\pi (\bar{x} - 1)/\bar{l}$$

$$\bar{\omega} = \cos my \sum_{n=1}^{N} W_n \sin n\pi (\bar{x} - 1)/\bar{l}$$

$$\bar{u} = u/a$$

$$\bar{v} = v/a$$

$$\bar{\omega} = \omega/a$$

$$\bar{l} = l/a$$

$$\bar{x} = x/a$$

The change of *potential energy in non-dimensional form*

$$\Delta \bar{U} = \Delta U / a^3 b_{11}$$

Then,

$$\Delta \bar{U} b_{11} = \frac{1}{2} \int_0^{2\pi} \int_1^{1+\bar{l}} \left\{ b_{11} \left(\bar{u}_{\bar{x}} \right)^2 + b_{22} \left(\frac{\bar{u}}{\bar{x}} + \frac{\bar{v}_y}{\bar{x} \sin\alpha} + \frac{\bar{\omega}}{\bar{x} \tan\alpha} \right)^2 \right.$$

$$+ b_{66} \left(\bar{v}_{\bar{x}} + \frac{\bar{u}_y}{\bar{x} \sin\alpha} - \frac{\bar{v}}{\bar{x}} \right)^2 + 2 b_{12} \bar{u}_{\bar{x}} \left(\frac{\bar{u}}{\bar{x}} + \frac{\bar{v}_y}{\bar{x} \sin\alpha} \right.$$

$$\left. \left. + \frac{\bar{\omega}}{\bar{x} \tan\alpha} \right) \right\} \bar{h} \bar{x} \sin\alpha \, d\bar{x} \, dy$$

$$+ \frac{1}{2} \int_0^{2\pi} \int_1^{1+\bar{l}} \left[\bar{N}_x (\bar{\omega}_{\bar{x}})^2 + \bar{N}_y \frac{(\bar{\omega}_y)^2}{\bar{x}^2 \sin^2\alpha} \right] \bar{x} \sin\alpha \, d\bar{x} \, dy$$

$$+ \frac{1}{24} \int_0^{2\pi} \int_1^{1+\bar{l}} \left[b_{11} \bar{\omega}_{xx}^2 + b_{66} \left(\frac{2\bar{\omega}_y}{\bar{x}^2 \sin\alpha} - \frac{2\bar{\omega}_{\bar{x}y}}{\bar{x} \sin\alpha} \right)^2 \right.$$

$$\left. + b_{22} \left(\frac{\bar{\omega}_{\bar{x}}}{\bar{x}} + \frac{\bar{\omega}_{yy}}{\bar{x}^2 \sin^2\alpha} \right)^2 + 2 b_{12} \bar{\omega}_{\bar{x}\bar{x}} \left(\frac{\bar{\omega}_{\bar{x}}}{\bar{x}} + \frac{\bar{\omega}_{yy}}{\bar{x}^2 \sin^2\alpha} \right) \right]$$

$$\cdot \bar{h}^3 \bar{x} \sin\alpha \, d\bar{x} \, dy \; .$$

Now substituting u, v and w in the above equation and minimizing the change of the potential energy expression with respect to U_p, V_p and W_p, an equation of the form as (22) is obtained, where,

$$A'_{np} = \left[\frac{b_{11}}{2\bar{l}} np\pi^2 (2+\bar{l}) + b_{22} \log(1+\bar{l}) + \frac{b_{66} m^2}{\sin^2\alpha} \right.$$

$$\left. \log(1+\bar{l}) \right] \delta_p^n + \alpha'_{np}$$

$$B'_{np} = - \left[b_{66} \frac{mn\pi}{\bar{l} \sin\alpha} + b_{12} \frac{mn\pi}{\bar{l} \sin\alpha} \right] \delta_p^n + \beta'_{np}$$

$$C'_{np} = -\frac{b_{12}\, n\pi}{\tan\alpha}\, \delta^n_p + \gamma'_{np}$$

$$A^2_{np} = B'_{pn}$$

$$B^2_{np} = \left[\frac{b_{22}\, m^2 \log(1+\bar{\ell})}{\sin\alpha \tan\alpha} + \frac{b_{66}\, p n\pi^2(\bar{\ell}+2)}{2\bar{\ell}} + b_{66} \log(1+\bar{\ell}) \right] \delta^n_p$$

$$C^2_{np} = \frac{b_{22}\, m \log(1+\bar{\ell})}{\sin\alpha \tan\alpha} + \gamma^2_{np}$$

$$A^3_{np} = C'_{pn}$$

$$B^3_{np} = C^2_{pn}$$

$$C^3_{np} = \left[\frac{b_{22}}{\tan^2\alpha} \log(1+\bar{\ell}) + \frac{b_{11}\, n^2\pi^4}{\bar{\ell}^3}\frac{(2+\bar{\ell})}{24}\bar{h}^2 p^2 + \frac{b_{66}(\bar{\ell}+2)\bar{h}^2}{6\sin^2\alpha(1+\bar{\ell})^2} \right.$$

$$+ \frac{b_{66}\, m^2 n\pi^2}{3\bar{\ell}^2 \sin^2\alpha} \log(\bar{\ell}+1)\bar{h}^2 p + \frac{p\bar{h}^2 b_{22}}{12\bar{\ell}^2} n\pi^2 \log(1+\bar{\ell})$$

$$\left. + \frac{b_{22}\, m^4 \bar{\ell}\,(\bar{\ell}+2a)\bar{h}^2}{24\sin^4\alpha\,(1+\bar{\ell})^2} + b_{12}\frac{m^2 n^2\pi^2\bar{h}^2}{6\bar{\ell}^2 \sin\alpha} \log(1+\bar{\ell}) \right] \delta^n_p + \gamma^3_{np}$$

$$\alpha'_{np} = \frac{b_{11}\, n p\pi^2}{\ell^2} \left[(\cos K_1\bar{\ell} - 1)/K_1^2 - (\cos K_2\bar{\ell} - 1)/K_2^2 \right]$$

$$+ b_{22} \left[F_1(K_1)\cos K_1 + F_2(K_1)\sin K_1 + F_1(K_2)\cos K_2 \right.$$

$$+ F_2(K_2) \sin K_2] + \frac{b_{66} m^2}{\sin^2 \alpha} \left[\cos K_1 F_1(K_1) + \sin K_1 F_2(K_1) \right.$$

$$\left. + \cos K_2 F_1(K_2) + \sin K_2 F_2(K_2) \right]$$

$$\beta'_{np} = \frac{b_{22} m}{\sin \alpha} \left[\cos K_2 F_2(K_2) - \sin K_2 F_1(K_2) - F_2(K_1) \cos K_1 \right.$$

$$\left. + \sin K_1 F_1(K_1) \right] - \frac{b_{66} n m \pi}{\ell \sin \alpha} \left[\cos K_1 F_1(K_1) + \sin K_1 F_2(K_1) \right.$$

$$\left. - \cos K_2 F_1(K_2) - \sin K_2 F_2(K_2) \right] - \frac{b_{12} m n \pi}{\ell \sin \alpha} \left[(\cos K_1 \ell - 1)/K_1 \right.$$

$$\left. - (\cos K_2 \ell - 1)/K_2 \right]$$

$$\gamma'_{np} = \frac{b_{22}}{\tan \alpha} \left[\cos K_2 F_2(K_2) - \sin K_2 F_1(K_2) - F_2(K_1) \cos K_1 \right.$$

$$\left. + F_1(K_1) \sin K_1 \right] - \frac{n \pi b_{12}}{\ell \tan \alpha} \left[(\cos K_1 \ell - 1)/K_1 \right.$$

$$\left. - (\cos K_2 \ell - 1)/K_2 \right]$$

$$\beta^2_{np} = \frac{m^2 b_{22}}{\sin^2 \alpha} \left[\cos K_1 F_1(K_1) \sin K_1 + F_2(K_1) \sin K_1 - \cos K_2 \right.$$

$$F_1(K_2) - \sin K_2 F_2(K_2) \right] + b_{66} \frac{n p \pi^2}{\ell^2} \left[(\cos K_1 \ell - 1)/K_1^2 \right.$$

$$\left. + (\cos K_2 \ell - 1)/K_2^2 \right] + b_{66} \left[\cos K_1 F_1(K_1) + \sin K_1 F_2(K_1) \right.$$

$$\left. - \cos K_2 F_1(K_2) - \sin K_2 F_2(K_2) \right] + \frac{b_{66} p}{\ell} \left[(\cos K_1 \ell - 1)/K_1 \right.$$

$$+ (\cos K_2 \bar{\ell} - 1)/K_2 \Big]$$

$$\gamma^2_{np} = \frac{b_{22} m}{\sin \alpha \tan \alpha} \Big[\cos K_1 \, F_1(K_1) + \sin K_1 \, F_2(K_1) - \cos K_2 \, F_1(K_2)$$

$$- \sin K_2 \, F_2(K_2) \Big]$$

$$\gamma^3_{np} = \frac{b_{22}}{\tan^2 \alpha} \Big[\cos K_1 \, F_1(K_1) + \sin K_1 \, F_2(K_1) - \cos K_2 \, F_1(K_2)$$

$$- \sin K_2 \, F_2(K_2) \Big] + \frac{\bar{h}^2}{12} \Big\{ \frac{b_{11} n^2 p^2 \pi^4}{\bar{\ell}^4} \Big[(\cos K_1 \bar{\ell} - 1)/k_1^2$$

$$- (\cos K_2 \bar{\ell} - 1)/K_2^2 \Big] + \frac{4 m^2 b_{66}}{\sin^2 \alpha} \Big[\cos K_1 \, F_5(K_1) +$$

$$\sin K_1 \, F_6(K_1) - \cos K_2 \, F_5(K_2) - \sin K_2 \, F_6(K_2) \Big]$$

$$- 4 b_{66} \frac{m^2 n \pi p}{\sin^2 \alpha \, \bar{\ell}^2} \Big[\cos K_1 \, F_1(K_1) + \sin K_1 \, F_2(K_1) +$$

$$\cos K_2 \, F_1(K_2) + \sin K_2 \, F_2(K_2) \Big] + \frac{4 \pi^2 m^2 n p}{\sin^2 \alpha} \Big[\cos K_1 \, F_1(K_1)$$

$$+ \sin K_1 \, F_2(K_1) + \cos K_2 \, F_1(K_2) + \sin K_2 \, F_2(K_2) \Big] +$$

$$\frac{\pi^2 n p}{\bar{\ell}^2} b_{22} \Big[\cos K_1 \, F_1(K_1) + \sin K_1 \, F_2(K_1) + \cos K_2 \, F_1(K_2)$$

$$+ \sin K_2 \, F_2(K_2) \Big] + \frac{n \pi \, b_{22} \, m^2}{\bar{\ell} \sin^3 \alpha} \Big[\cos K_2 \, F_2(K_2) - \sin K_2 \, F_1(K_2)$$

$$-\cos K_1 F_2(K_1) + \sin K_1 F_1(K_1)\Big] + \frac{b_{22}m^4}{\sin^4\alpha}\Big[\cos K_1 F_5(K_1)$$

$$+ \sin K_1 F_6(K_1) - \cos K_2 F_5(K_2) - \sin K_2 F_6(K_2)\Big]$$

$$+ \frac{b_{12}\pi^2 n^2 m^2}{\bar{l}^2 \sin^2\alpha}\Big[\cos K_1 F_1(K_1) + \sin K_1 F_2(K_1) + \cos K_2 F_1(K_2)$$

$$+ \sin K_2 F_2(K_2)\Big]\Big\}$$

where,

$$F_1(z) = \Big[\log x - \frac{z^2\bar{x}^2}{2\cdot 2!} + \frac{z^4\bar{x}^4}{4\cdot 4!} - \frac{z^6\bar{x}^6}{6\cdot 6!} + \cdots \Big]_{\bar{x}=1}^{\bar{x}=1+\bar{l}}$$

$$F_2(z) = \Big[z\bar{x} - \frac{z^3\bar{x}^3}{3\cdot 3!} + \frac{z^5\bar{x}^5}{5\cdot 5!} - \frac{z^7\bar{x}^7}{7\cdot 7!} + \cdots \Big]_{\bar{x}=1}^{\bar{x}=1+\bar{l}}$$

$$F_5(z) = \Big[-\frac{3}{\bar{z}^2} + \frac{z^2\log\bar{x}}{2!} + \frac{z^4\bar{x}^2}{2\cdot 4!} - \frac{z^6\bar{x}^6}{6\cdot 6!} + \cdots \Big]_{\bar{x}=1}^{\bar{x}=1+\bar{l}}$$

$$F_6(z) = \Big[-\frac{2z}{\bar{x}} - \frac{z^2\bar{x}}{3!} + \frac{z^5 x^3}{3\cdot 5!} - \frac{z^7 x^7}{5\cdot 7!} + \cdots \Big]_{\bar{x}=1}^{\bar{x}=1+\bar{l}}$$

$$F_9(z) = \left[-\frac{2}{\bar{x}} - \frac{z^2\bar{x}}{2!} + \frac{z^4\bar{x}^3}{3\cdot 4!} - \frac{z^6 x^5}{5\cdot 6!} + \cdots \right]_{\bar{x}=1}^{\bar{x}=1+\bar{l}}$$

$$F_{10}(z) = \left[z\log\bar{x} - \frac{z^2\bar{x}^2}{2\cdot 3!} + \frac{z^5 x^4}{4\cdot 5!} - \frac{z^7 x^6}{6\cdot 7!} + \right]_{\bar{x}=1}^{\bar{x}=1+\bar{l}}$$

$$K_1 = (n-n')\pi/\bar{l}$$

$$K_2 = (n+n')\pi/\bar{l}$$

If only symmetric modes are considered then

$$\alpha'_{np} = \beta'_{np} = \beta^2_{np} = \gamma'_{np} = \gamma^2_{np} = \gamma^3_{np} = 0$$

Section VII

SPECIAL PROBLEMS AND NUMERICAL EXAMPLES

VII.1 Thermal Co-efficients of Composite

For the fiber and the matrix properties given in
Table 8.1, bounds on the thermal co-efficients are obtained
using the inequality developed in Section III. A computer
program is set up for finding the bounds. For obtaining
the elastic and the thermal constants of transversely
isotropic phases, methods suggested by Whitney et al (6)
and Schapery (12) have been used. The upper bound of the
thermal co-efficients, which will be of more importance
in the thermal stress and the thermal buckling problems,
are found to be,

$$\alpha_{x\text{-direction}} \leq 1.65 \times 10^{-5}$$

$$\alpha_{y\text{-direction}} \leq 1.60 \times 10^{-5}$$

The above result is for the composite with 50% fiber
content.

VII.2 Thermal Stresses in Shells

For the problem of thermal stresses and thermal
buckling six different types of orthotropic material
(see Table 8.2) will be considered. Case No. 1 in the

Table 8.2 corresponds to a special type of orthotropy which is isotropy.

Cylindrical Shell

(I) Simple Supported Case

(A) First a temperature distribution varying through the thickness is considered. Since the thickness of the shell is small, the temperature distribution through the thickness can be reasonably approximated by a linear distribution,

$$T = T_0 + T_i \, z/a \tag{1}$$

For a temperature distribution given by equation (1) the displacement and the stress components will be obtained from two different sets of equilibrium equations (See equation (14) and (23), Section V).

For the particular temperature distribution, f( ) from equation (17), Section V, will be,

$$f(\theta) = \frac{144 \, a^2}{h^2 (12a^2 - h^2)} \left[\frac{b_{12}}{b_{11}^2 a} C_1 \left(T_0 + \frac{T_i h^2}{12 \, a^2} \right) - C_2 \frac{T_0}{a b_{11}} \right]$$

$$= K \; (\text{say}) \tag{2}$$

Substituting the value of $f(\theta)$ in the equation (14), Section V, the particular solution W_p of the equation

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is found to be

$$W_p = K/4\beta^4 \qquad (3)$$

Therefore, the general solution of the equation (14), Section V, is given by,

$$\omega = e^{-K_1 x}\left[B_1 \cos K_2 x + B_2 \sin K_2 x\right] + e^{K_1 x}\left[B_3 \cos K_2 x + B_4 \sin K_2 x\right] + K/4\beta^4 \qquad (4)$$

where,

$$K_1 = \sqrt{\beta^2 - \alpha^2}$$

$$K_2 = \sqrt{\beta^2 + \alpha^2} \qquad (5)$$

For the simple supported case, the boundary conditions are as follows.

$$\omega = 0$$
$$M_x = 0 \qquad @ \quad x = 0, \ell \qquad (6)$$

From the boundary condition on displacements,

$$B_1 + B_3 = -K/4\beta^4 \tag{7}$$

$$e^{-K_1 l}\left[B_1 \cos K_2 l + B_2 \sin K_2 l\right] + e^{K_1 l}\left[B_3 \cos K_2 l + B_4 \sin K_2 l\right]$$
$$= -K/4\beta^4 \tag{8}$$

To apply the boundary conditions on moments, $\bar{u}_x$ has to be eliminated from the expression of M'_x by means of equation (13), Section V. The moment boundary condition then becomes,

$$\frac{b_{11} h^3}{12}\left[\frac{h^2}{12 a^2} - 1\right]\omega_{xx} + \frac{h^2}{12 a}\int_{-h/2}^{h/2}\left[C_1\theta\left(1 + z/a\right)\right]dz$$
$$- \int_{-h/2}^{h/2} C_1\theta\left(z + z^2/a\right)dz = 0 \tag{9}$$

Applying the above two boundary conditions following equations are obtained,

$$P(K_1^2 - K_2^2)(B_1 + B_3) + 2 K_1 K_2 P(B_4 - B_2) = \frac{C_1 h^3}{12 a}\left(T_0 + T_1\right) - \frac{h^3}{12 a}C_1\left(T_0 + \frac{T_1 h^2}{12 a^2}\right) \tag{10}$$

$$B_1\left[(K_1^2 - K_2^2)\cos K_2 l + 2 K_1 K_2 \sin K_2 l\right]e^{-K_1 l} + B_2\left[(K_1^2 - K_2^2)\sin K_2 l\right.$$
$$\left. - 2 K_1 K_2 \cos K_2 l\right]e^{-K_1 l} + B_3\left[(K_1^2 - K_2^2)\cos K_2 l - 2 K_1 K_2 \sin K_2 l\right]e^{K_1 l}$$
$$+ B_4\left[(K_1^2 - K_2^2)\sin K_2 l + 2 K_1 K_2 \cos K_2 l\right]e^{K_1 l} =$$
$$\left\{\frac{C_1 h^3}{12 a}\left(T_0 + T_1\right)\right.$$

$$-\frac{h^3}{12a} \left(T_0 + \frac{T_1 h^2}{12 a^2} \right) \Big\} \Big/ \rho \qquad (11)$$

Where,

$$\rho = \frac{b_{11} h^3}{12} \left(\frac{h^2}{12 a^2} - 1 \right) \qquad (12)$$

Equations (7), (8), (10) and (11) are solved for B_1, B_2, B_3, B_4. The radial displacement w for a shell of 100" long, 100" diameter and .50" thick is tabulated in Table 8.3 (Case 2). The particular temperature distribution considered is T=50+1000Z.

To solve the same problem by means of simplified sets of equilibrium equations (Equations (25), and (30), Section V), it is noted that particular integral W_p is given by

$$W_p = A_2 T_0 h / 4 \kappa^4 \qquad (13)$$

From the natural boundary condition (21), Section V and the forced boundary condition w=o @ $x = 0, \ell$ following equations are obtained.

$$B_4 - B_1 = - c_1 T_1 / 2\kappa^2 b_{11} a \qquad (14)$$

$$(B_1 \sin \kappa \ell - B_2 \cos \kappa \ell) e^{-\kappa \ell} + (B_4 \cos \kappa \ell - B_3 \sin \kappa \ell) e^{\kappa \ell}$$
$$= - c_1 T_1 / 2 \kappa^2 b_{11} a \qquad (15)$$

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$$B_1 + B_3 = -A_2 T_0 h / 4 K^4 \tag{16}$$

$$e^{-K\ell}\left[B_1 \cos K\ell + B_2 \sin K\ell\right] + e^{K\ell}\left[B_3 \cos K\ell + B_4 \sin K\ell\right]$$

$$= -A_2 T_0 h / 4 K^4 \tag{17}$$

From equations (14), (15), (16) and (17), B_1, B_2, B_3 and B_4 can be determined. They are given by,

$$B_4 = \left[\frac{(E_4 - D_1 E_3 + D_2 E_1)}{D_3 - D_1} + \frac{(E_2 - D_2 E_3 - D_1 E_1)}{D_2 + D_4} \right] \bigg/ \left(\frac{D_3 - D_1}{D_2 + D_4} + \frac{D_2 + D_4}{D_3 - D_1} \right) \tag{18}$$

$$B_3 = \frac{E_4 - D_1 E_3 + D_2 E_1}{D_3 - D_1} - \frac{B_4 (D_2 + D_4)}{(D_3 - D_1)} \tag{19}$$

$$B_2 = B_4 - E_1 \tag{20}$$

$$B_1 = E_3 - B_3 \tag{21}$$

where, E_1, E_2, E_3 and E_4 are right hand side of the equations (14), (15), (16) and (17).

$$D_1 = e^{-K\ell} \cos K\ell$$

$$D_2 = e^{-K\ell} \sin K\ell$$

$$D_3 = e^{K\ell} \cos K\ell$$

$$D_4 = e^{K\ell} \sin K\ell \tag{22}$$

$\bar{u}_x$ is obtained from equation (25), the stresses will be given by,

$$\sigma_{\bar{x}} = 0$$

$$\sigma_{\bar{y}} = (\omega - \omega_p)\left(\frac{b_{12}}{a} - \frac{b_{12}^2}{b_{11} a}\right)$$

$$\tau_{xy} = 0 \tag{23}$$

The hoopstresses obtained for different cases of orthotropy is shown in Figure 8.1. The dimensions of the shell is as before. The displacement component w is tabulated in Table -8.3 (Case 1).

(B) Temperature Varying in Axial Direction

A temperature distribution of the following form is considered.

$$T = T_o \, e^{-x/a} \tag{24}$$

The particular integral in that case is given by,

$$\omega_p = S \, e^{-x/a}$$

where

$$S = \frac{A_2 T_o h}{\left(1/a^4 + 4 K^4\right)} \tag{25}$$

E_1, E_2, E_3 and E_4 are given by,

$$E_1 = - A_2 T_0 h \, a^2 \big/ 2 \kappa^2 (1 + 4 a^4 \kappa^4)$$

$$E_2 = E_1 e^{-\ell/a}$$

$$E_3 = - A_2 T_0 h \big/ (1/a^4 + 4\kappa^4)$$

$$E_4 = E_3 e^{-\ell/a}$$

(26)

The hoopstress developed for such temperature distribution are shown in Figure 8.2.

(C) Temperature Varying along Generator as Well as Thickness

A temperature distribution of the following form is considered:

$$T = T_0 e^{-x/a} (1 + T_1 z/a)$$ (27)

The particular integral in this case

$$W_P = S e^{-x/a}$$ (28)

where,

$$S = T_0 \left(\frac{A_1 T_1 h^3}{12 \, a^3} + A_2 h \right) \Big/ (1/a^4 + 4\kappa^4)$$ (29)

E_1, E_2 etc. are given by,

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$$E_{11} = -12S / (2 \kappa^2 a^2 h^3 b_{11}) - c_1 T_0 T_1 / b_{11} a$$

$$E_2 = E_1 e^{-l/a}$$

$$E_3 = -S$$

$$E_4 = -S e^{-l/a} \qquad\qquad (30)$$

(II)   Fixed End Case

For a temperature distribution as given by equation (1), the particular integral is given by,

$$W_p = A_2 T_c h / 4 \kappa^4 - 3 \bar{K} b_{12} / a^2 b_{11} \kappa^4 h^2 b_{11} \qquad (31)$$

To evaluate the constants B_1, B_2, B_3 and B_4 the following force boundary conditions are to be applied.

$$\omega = \omega_x = 0 \qquad\qquad @ \quad x = 0, l \qquad (32)$$

To evaluate $\bar{K}$, the equation (24), Section V, is integrated from o to 1 and the boundary condition (31) is utilized and the following equation for $\bar{K}$ is obtained.

$$\bar{K} = F_1 / F_2 \qquad\qquad (33)$$

where,

$$F_1 = b_{12} \left[B_1' \frac{e^{-K\ell}}{2K^2} \left(K \sin K\ell - K \cos K\ell \right) - \frac{B_2' e^{-K\ell}}{2K^2} \left(K \cos K\ell + K \sin K\ell \right) \right.$$

$$+ B_3' \frac{e^{K\ell}}{2K^2} \left(K \cos K\ell + K \sin K\ell \right) + B_4' \frac{e^{K\ell}}{2K^2} \left(K \sin K\ell - K \cos K\ell \right)$$

$$\left. - \frac{1}{2K} \left(B_3' + B_4' - B_1' - B_2' \right) \right] - \int_0^\ell a c_1 \theta_0 / h \, dx \qquad (34)$$

$$F_2 = \ell - b_{12} \left[B_1'' \frac{e^{-K\ell}}{2K^2} \left(K \sin K\ell - K \cos K\ell \right) - \frac{B_2''}{2K^2} \left(K \cos K\ell + K \sin K\ell \right) \right.$$

$$+ B_3'' \frac{e^{K\ell}}{2K^2} \left(K \cos K\ell + K \sin K\ell \right) + \frac{B_4''}{2K^2} \left(K \sin K\ell - K \cos K\ell \right)$$

$$\left. - \frac{1}{2K} \left(B_3'' + B_4'' - B_1'' - B_2'' \right) \right. \qquad (35)$$

$$B_i = B_i' + B_i'' \bar{K} \qquad (36)$$

The stresses are given by the following equation.

$$\sigma_x = \bar{K}/a + \left(c_1 \theta_0 / h - c_1 \theta_m \right)$$

$$\sigma_y = \left(\omega - \omega_p \right) \left(b_{22}/a - b_{12}^2 / b_{11} a \right)$$

$$\tau_{xy} = 0 \qquad (37)$$

VII.3 Thermal Buckling of Shells

For the simple supported case, only three modes are considered to be present. Calculations for few cases showed that if four modes are considered the values of critical temperature differ only by 1 to 3 percent. For different values of thickness, length and temperature gradient, critical temperatures have been found and

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results are presented in Figures 8.3 through 8.7.

For the case of fixed end, a two mode solution is considered. For the particular case, two-mode solution is found to be sufficiently accurate. Digital computer is used for all calculations and the results are presented in Figures 8.8 and 8.9. For fixed end case, only uniform temperature rise has been considered.

Section VIII

DISCUSSIONS AND CONCLUSIONS

The expression of effective elastic constants developed in Section II are mainly due to Paul (7). In Section III effective thermal constants for bi-directionally reinforced composite have been developed. For the particular case considered in the last section, the numerical values of thermal co-efficients obtained, appear reasonable.

In Section V, two sets of equilibrium equations have been derived both for the case of cylindrical shells and conical shells. One set of equilibrium equation is derived from general equilibrium equation given by Love (15). The other set is obtained by setting the first variation of approximate strain energy expression equal to zero. For cylindrical shells, Table 8.3 shows the radial displacements calculated from two different sets of equilibrium equations differ by less than 8%.

For the simple supported case, the effect of thermal co-efficients on the hoopstress is worth noting. The hoopstress is more sensitive to change of the thermal co-efficient in circumferential direction than the change of that in axial direction. Figure 8.1 shows that the maximum hoopstress is decreased by about 71.4% from the

isotropic case (Case No. 1) when the thermal co-efficient
in circumferential direction is reduced to 50% of its
isotropic value (Case No. 2). The maximum hoopstress,
on the other hand, increases about 21.4% from the isotropic
value, when the thermal co-efficient in axial direction
is reduced by 50%. The stiffness ratio does not have
any effect on maximum hoopstress in the particular case
considered.

The critical temperature for the case of isotropic
shell subjected to uniform temperature rise agrees with
result given by Hoff(21). Figure 8.3 through 8.7 shows
the critical temperature against various parameters for
the simple supported case.

Figure 8.3 shows the effect of L/a ratio on the
critical temperature of the shell for a radius-thickness
ratio of 2000, when the shell is subjected to uniform rise
of temperature. In the Figure 8.4, the effect of a/h ratio
on critical temperature, for a length-radius ratio of .05,
are shown. The effect of the temperature gradient on
critical temperatures are shown in Figure 8.5. In the
particular case considered, the radius-thickness ratio is
2000 and length-radius ratio is .05.

From these figures it is apparent, that in simple
supported cases, for thermal buckling to occur, the
shell has to be very thin and short. For a thick or long

shells, the calculated critical temperature will be so high, that the stresses at the edges will exceed elastic limit and the plastic deformation will take place before elastic buckling can occur.

Figure 8.3 shows that with the increase in length the critical temperature decreases at the beginning and after reaching a minimum value it goes on increasing. When the length of the shell decreases, compressive hoop-stresses which are developed because of edge constraints, tend to spread over the entire length. The effect of this phenomenon is to reduce the critical temperature. On the other hand, a decrease in length increases the stiffness of the shell and thereby tend to increase the critical temperature. The presence of the above phenomena explains the nature critical temperature curve in the Figure 8.3.

It is interesting to note that a 50% reduction in the thermal co-efficient in axial direction reduces the buckling temperature by approximately 18% from that of isotropic case. This is, of course, the consequence of the earlier observation that hoopstress increases with the decrease in thermal co-efficient in axial direction.

A 50% reduction in thermal co-efficient in circum-ferential direction, on the other hand, increases buckling temperature by about 250%. Here a point should be noted.

Since in the present analysis, it has been assumed that the elastic and the thermal properties are independent of temperature, this analysis should give fairly accurate results when temperature is not too high. For the above case, since the critical temperature is too high, its value is questionable.

The stiffness ratio plays an important role on the critical temperature. The decrease in elastic co-efficient in circumferential direction, decreases the critical temperature. Within the range of .05 to .2 of l/a ratio, the critical temperature is reduced by 50-56% from the isotropic critical temperature when the elastic constant in circumferential direction is reduced by 50%. A 100% increase in elastic constant in circumferential direction increases the critical temperature up to 169%, in the range considered. Therefore such an increase in elastic constants considerably reduces the possibility of thermal buckling. In fact for the above increment in stiffness ratio, the buckling will occur only within the range where the critical temperature curve dips. For any other length plastic deformation will occur, at the edges.

Figure 8.5 indicates that the presence of thermal gradient, with temperature higher towards inner surface,

reduces the critical temperature. This is because of the development of thermal moments at the supported end.

Figures 8.6 and 8.7 correspond to a temperature distribution of the form $T=T_\theta e^{-x/a}$. For such an axially varying temperature distribution, the critical temperature $T_{0_{cr}}$ is little higher than that in the case of uniform temperature rise.

Figures 8.8 shows the effect of the length-radius ratio on the critical temperature when the ends of the shell are fixed. For this type of boundary condition, the buckling occurs mainly because of thermal compression. The length of the shell does not, therefore, has much effect on the critical temperature, when the length is not very small. For short shells the effect of hoopstress will be noticeable. Figure 8.8 shows that the critical temperature decreases first with the increase in length, then it increases a little and after that it remains constant.

Figure 8.9 shows that critical temperature monotonically decreases with the increase in radius-thickness ratio in case of fixed end shells.

No numerical result for conical shells has been presented. It may be noted that buckling criteria for conical shell takes simple form when only the symmetric modes

are present. Since, only small cone angle are considered in the study and buckling occurs only for small value of l/a ratio, the calculation of critical temperature on the basis of symmetric buckling will give fairly accurate value, for the cases considered in cylindrical shell.

Based on our foregoing discussion the following conclusions can be made.

(1) For simple supported case where the edges are restrained in circumferential direction but are free to move in axial direction, the thermal buckling will occur only for very thin and short shells.

(2) A decrease in thermal co-efficients in axial direction, increases the possibility of thermal buckling. Quantitatively, a 50% decrease in thermal co-efficient reduces critical temperatures by 18% from its value in isotropic case.

(3) A decrease in thermal co-efficient in circumferential direction reduces the possibility of thermal buckling.

(4) An increase in elastic constants in circumferential direction considerably

reduces the thermal buckling possibility.

(5) A negative thermal gradient (higher
temperature towards inner surface) reduces
the critical temperature and thereby makes
the shell more susceptible to buckling.

(6) For fixed end case thermal buckling may
occur for moderately thin shells.

(7) The critical temperature for fixed end case
is independent of the shell length except
for short shells where hoopstresses developed
affect the critical temperature to a small
extent. In case of long shells, buckling
occurs due to thermal compression and the
shell may be expected to buckle into multi-
ple waves.

(8) The general buckling criterian that has been
derived contain pre-buckling rotation terms.
For numerical results, buckling criterian
without pre-buckling rotation terms has been
considered. Pre-buckling rotation terms may
affect the buckling temperature to certain
extent and this remains to be investigated.

In this study a buckling criterian that is
applicable to an arbitrary temperature distribution
has been developed. However numerical result presented
here is only for axi-symmetric case.

Table 8.1

Matrix	Fiber (X-direction)	Fiber (Y-direction)
$\alpha_1 = 6 \times 10^{-6}/^{\circ}C$	$\alpha_2 = 3 \times 10^{-5}/^{\circ}C$	$\alpha_3 = 0.49 \times 10^{-5}/^{\circ}C$
$E_1 = 0.44 \times 10^{+6} psii$	$E_2 = 2 \times 10^6 psi$	$E_3 = 69.8 \times 10^6 psi$
$\nu = 0.382$	$\nu = 0.3$	$\nu = 0.3$

Table 8.2

Case No	b_{12}/b_{11}	b_{22}/b_{11}	C_1/b_{11} per$^{\circ}$F	C_2/b_{11} per$^{\circ}$F
1	0.3	1.0	10^{-5}	10^{-5}
2	0.3	1.0	10^{-5}	$.5 \times 10^{-5}$
3	0.3	1.0	$.5 \times 10^{-5}$	10^{-5}
4	0.3	0.5	10^{-5}	10^{-5}
5	0.3	2.0	10^{-5}	10^{-5}
6	0.3	2.0	$.5 \times 10^{-5}$	10^{-5}

TABLE 8.3

Axial Distance	Radial Displ.[1]		Radial Displ.[2]		Radial Displ.[3]		Radial Displ.[4]	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0	0	0	0	0	0	0	0	0
5	$.1814 \times 10^{-1}$	$.1774 \times 10^{-1}$	$.5468 \times 10^{-2}$	$.5066 \times 10^{-2}$	$.2173.10^{-1}$	$.2153.10^{-1}$	$.3600.10^{-1}$	$.3531.10^{-1}$
10	$.2053 \times 10^{-1}$	$.2047 \times 10^{-1}$	$.5911 \times 10^{-2}$	$.5847 \times 10^{-2}$	$.2489.10^{-1}$	$.2485.10^{-1}$	$.4557.10^{-1}$	$.4534.10^{-1}$
15	$.1952 \times 10^{-1}$	$.1953.10^{-1}$	$.5561.10^{-2}$	$.5581 \times 10^{-2}$	$.2371.10^{-1}$	$.2372.10^{-1}$	$.4449.10^{-1}$	$.4449.10^{-1}$
20	$.1918 \times 10^{-1}$	$.1918 \times 10^{-1}$	$.5473 \times 10^{-2}$	$.5479 \times 10^{-2}$	$.2329.10^{-1}$	$.2278.10^{-1}$	$.4296.10^{-1}$	$.4298.10^{-1}$
25	$.1920 \times 10^{-1}$	$.1923 \times 10^{-1}$	$.5486 \times 10^{-2}$	$.5485 \times 10^{-2}$	$.2331.10^{-1}$	$.2297.10^{-1}$	$.4256.10^{-1}$	$.4256.10^{-1}$
30	$.1923 \times 10^{-1}$	$.1923 \times 10^{-1}$	$.5495 \times 10^{-2}$	$.5493 \times 10^{-2}$	$.2335.10^{-1}$	$.2334.10^{-1}$	$.4261.10^{-1}$	$.4267.10^{-1}$
35	$.1923 \times 10^{-1}$	$.1923 \times 10^{-1}$	$.5495 \times 10^{-2}$	$.5493 \times 10^{-2}$	$.2335.10^{-1}$	$.2336.10^{-1}$	$.4269 \times 10^{-1}$	$.4267.10^{-1}$

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     of a Thin Walled Circular Cylindrical Shell Heated
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Table 8.1

Matrix	Fiber (X-direction)	Fiber (Y-direction)
$\alpha_1 = 6 \times 10^{-6}/^\circ C$	$\alpha_2 = 3 \times 10^{-5}/^\circ C$	$\alpha_3 = 0.49 \times 10^{-5}/^\circ C$
$E_1 = 0.44 \times 10^{+6}$ psi	$E_2 = 2 \times 10^6$ psi	$E_3 = 69.8 \times 10^6$ psi
$\nu = 0.382$	$\nu = 0.3$	$\nu = 0.3$

Table 8.2

Case No	b_{12}/b_{11}	b_{22}/b_{11}	C_1/b_{11} per°F	C_2/b_{11} per°F
1	0.3	1.0	10^{-5}	10^{-5}
2	0.3	1.0	10^{-5}	$.5 \times 10^{-5}$
3	0.3	1.0	$.5 \times 10^{-5}$	10^{-5}
4	0.3	0.5	10^{-5}	10^{-5}
5	0.3	2.0	10^{-5}	10^{-5}
6	0.3	2.0	$.5 \times 10^{-5}$	10^{-5}

FIGURE 2.1
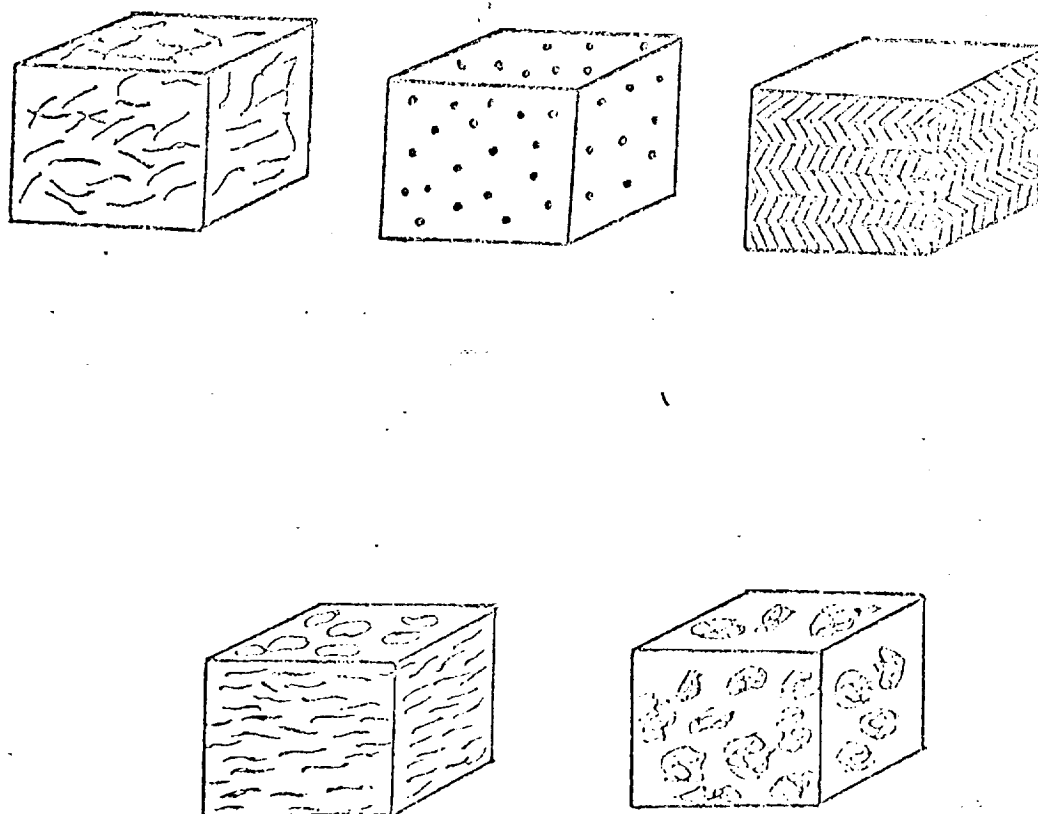
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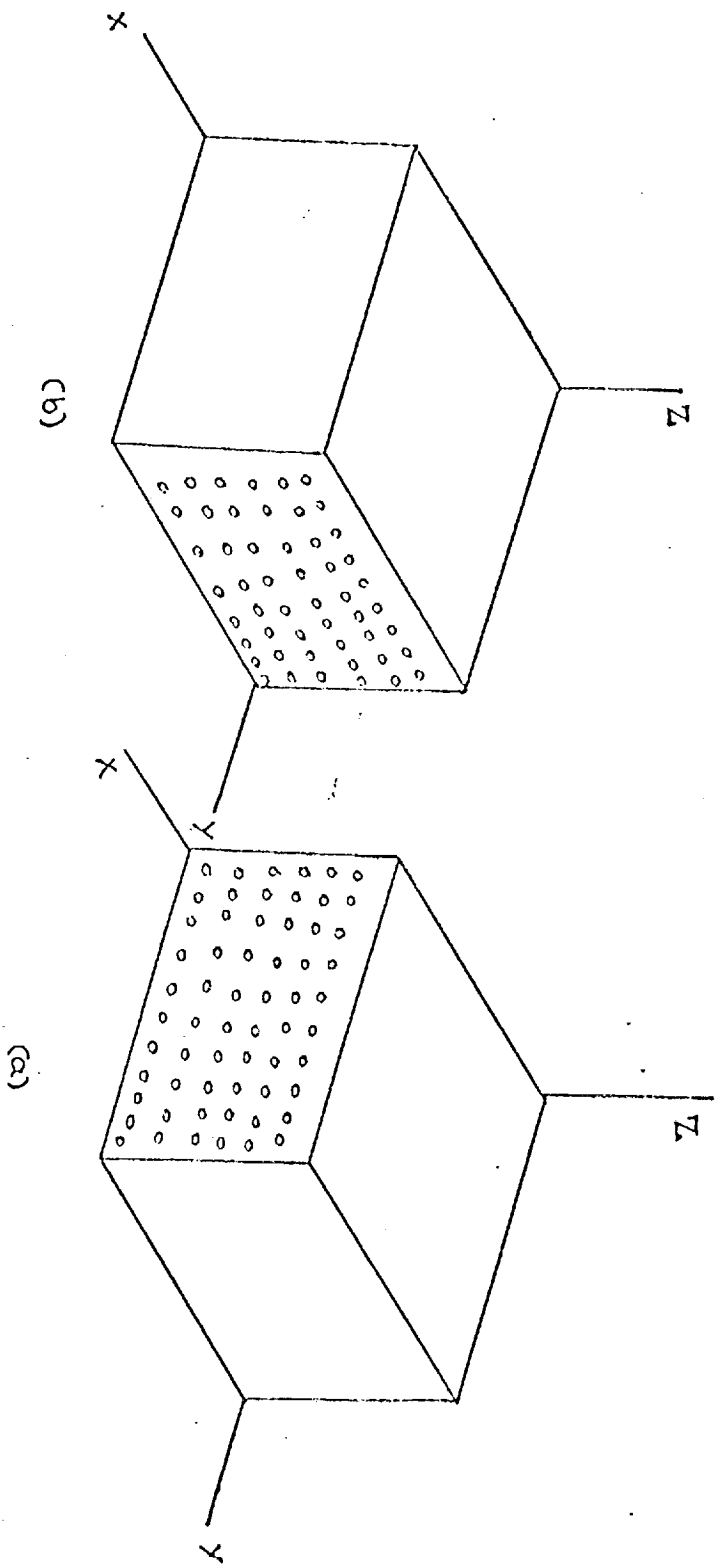
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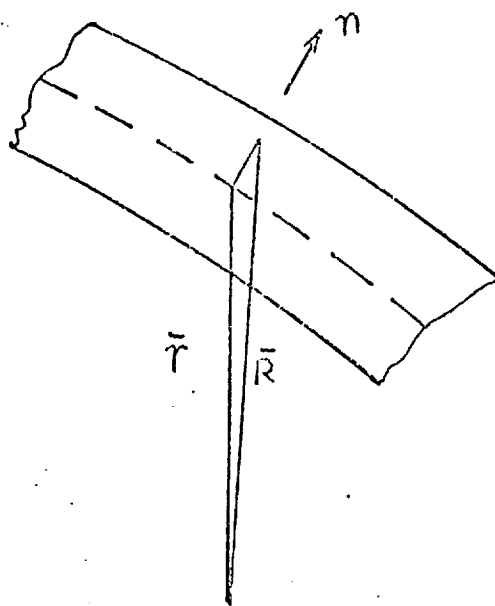
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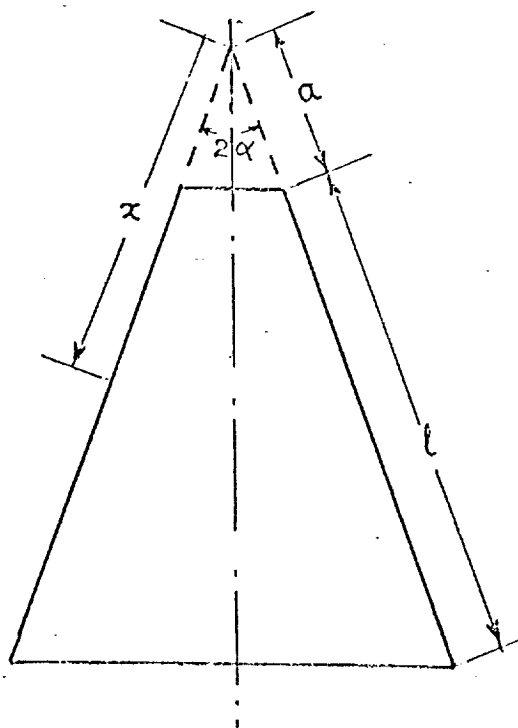
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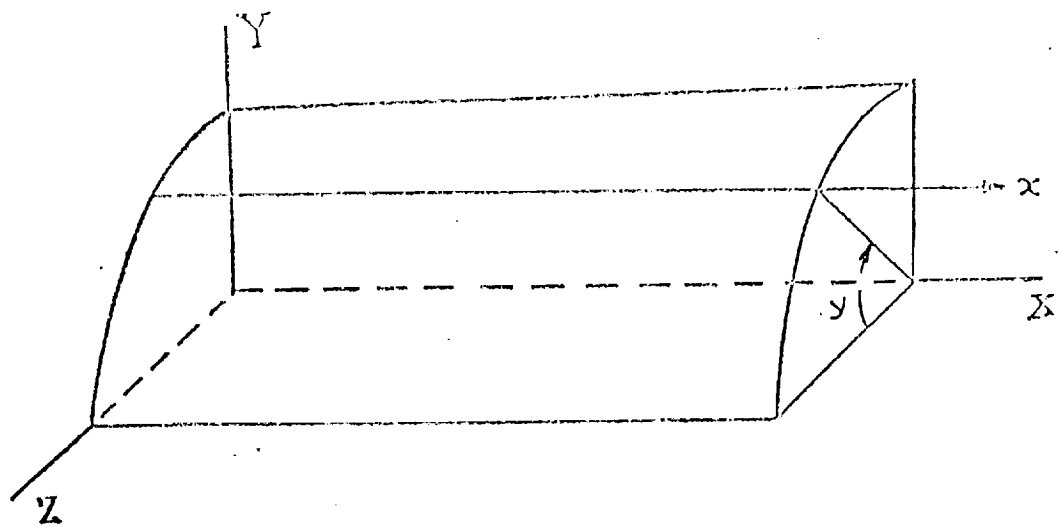
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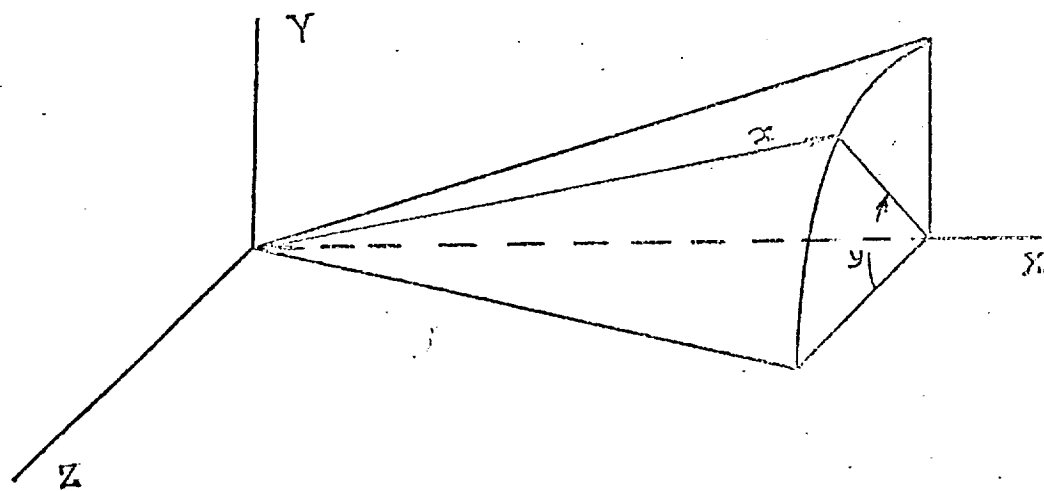
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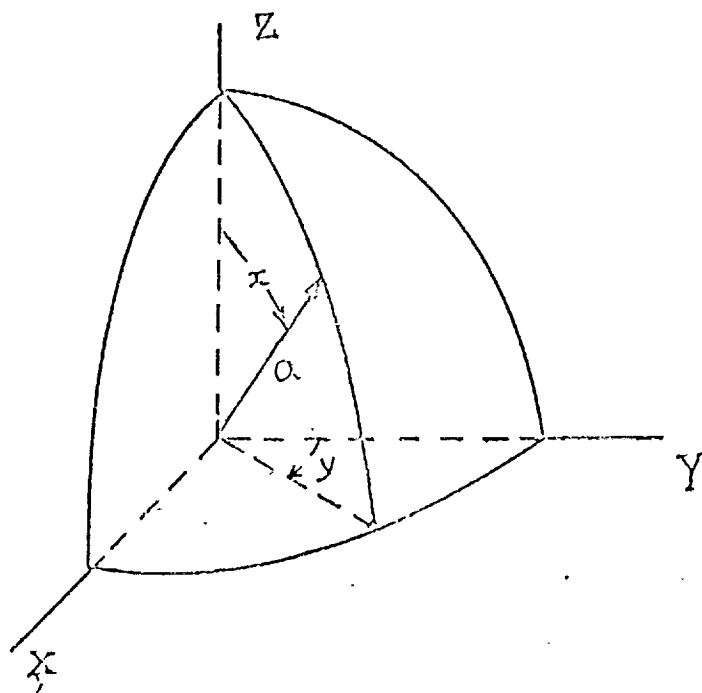
FIGURE 4.4

FIGURE 5.1

$$10^6 s_Y/B_{11}T$$

AXIAL DISTANCE FROM ONE END   inches

2.0   0   -2.0   -4.0   -6.0   -8.0   -10.0
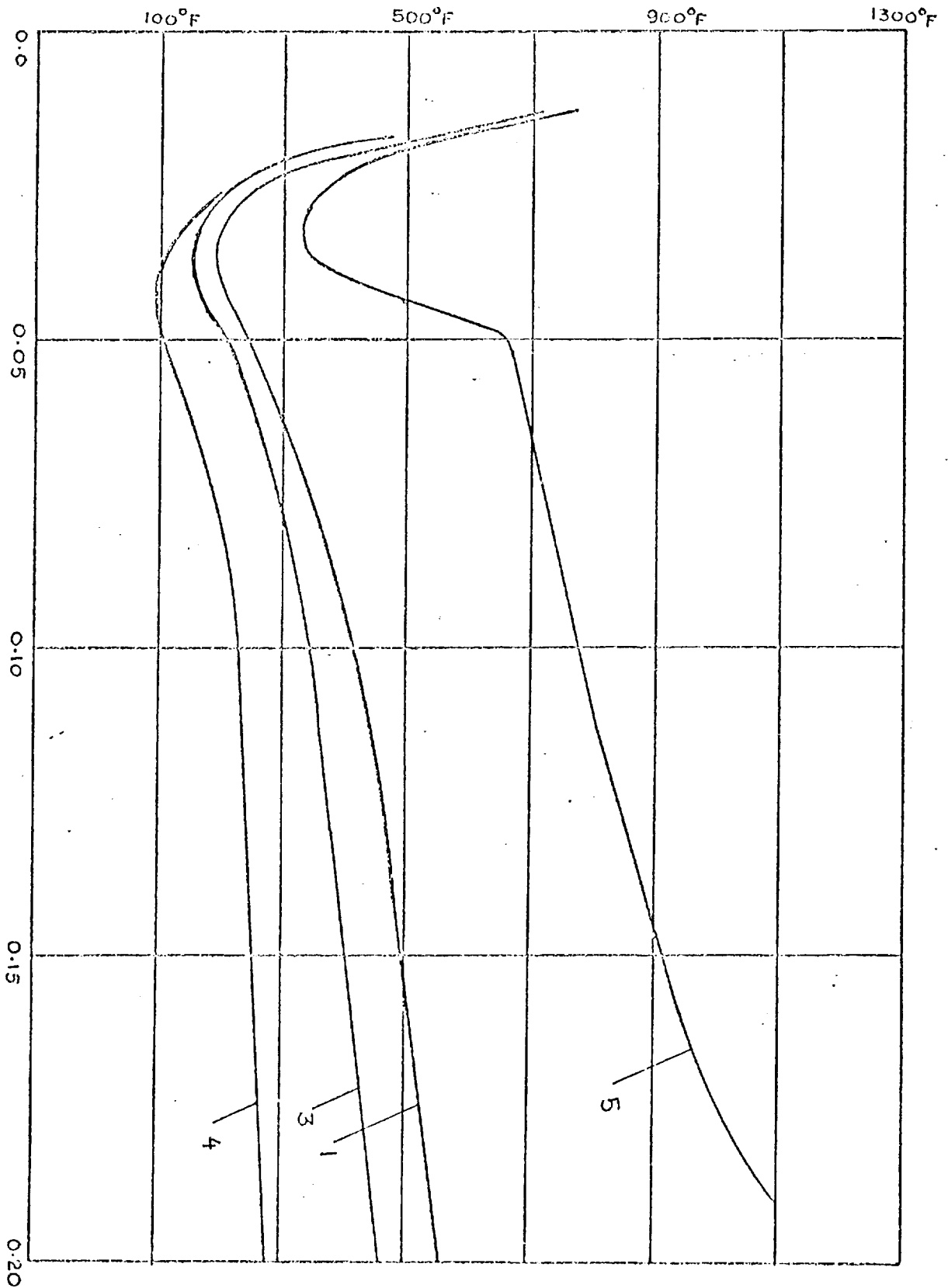
5   10   15

2
5
1
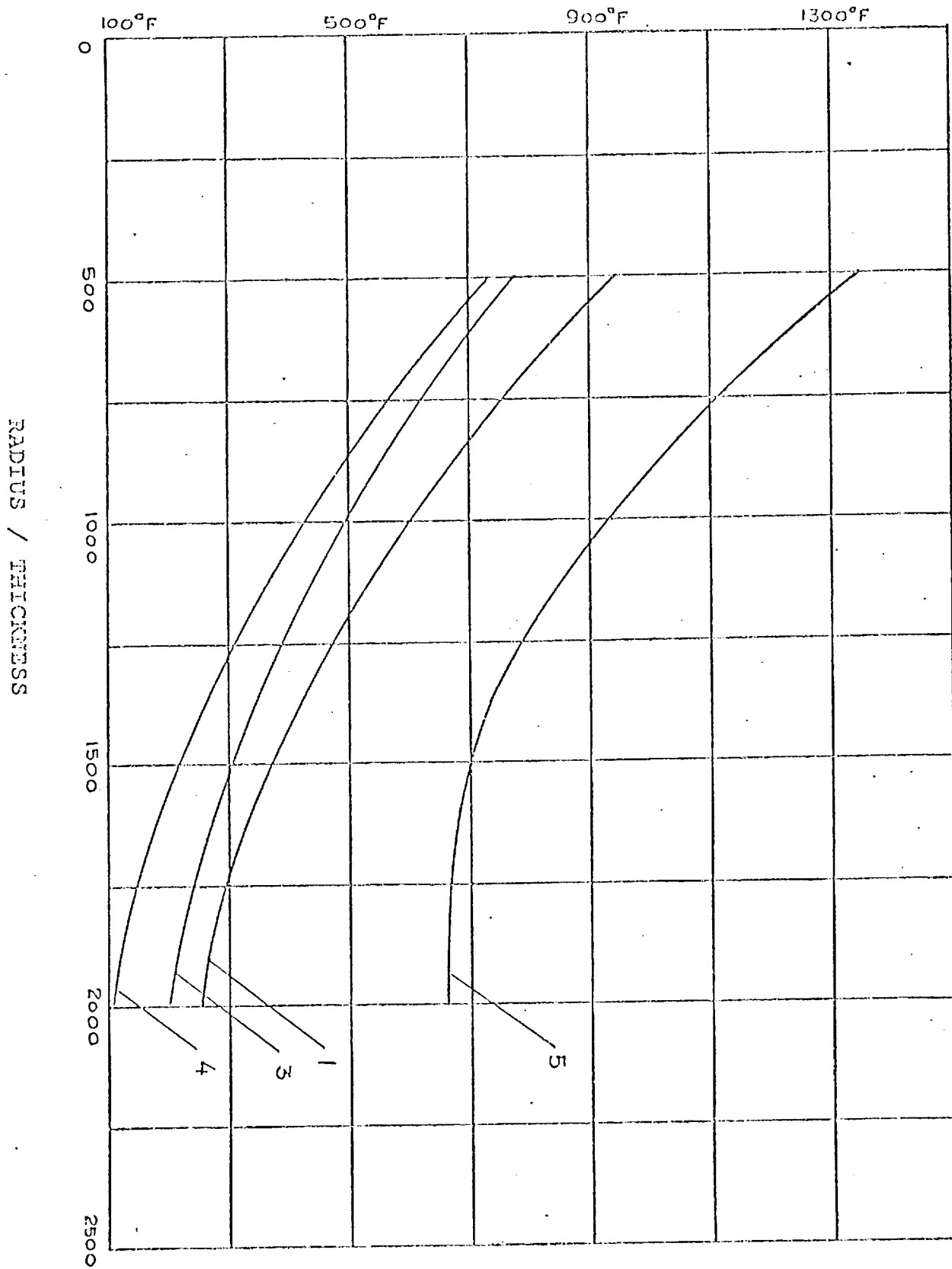3
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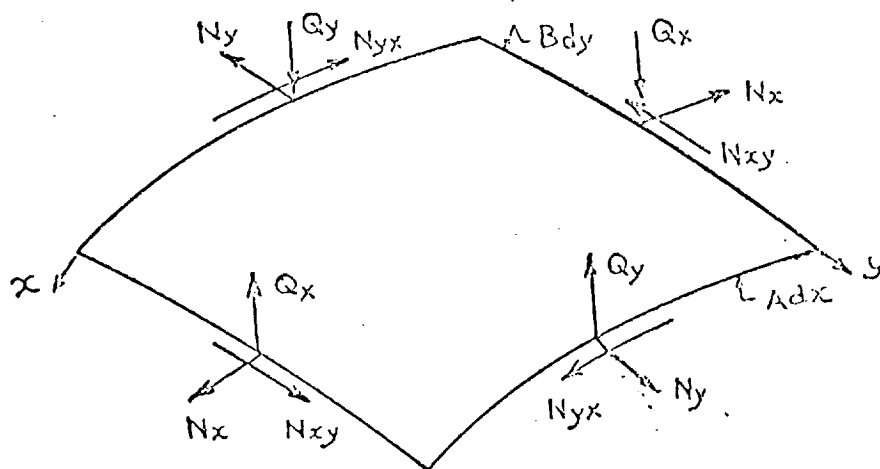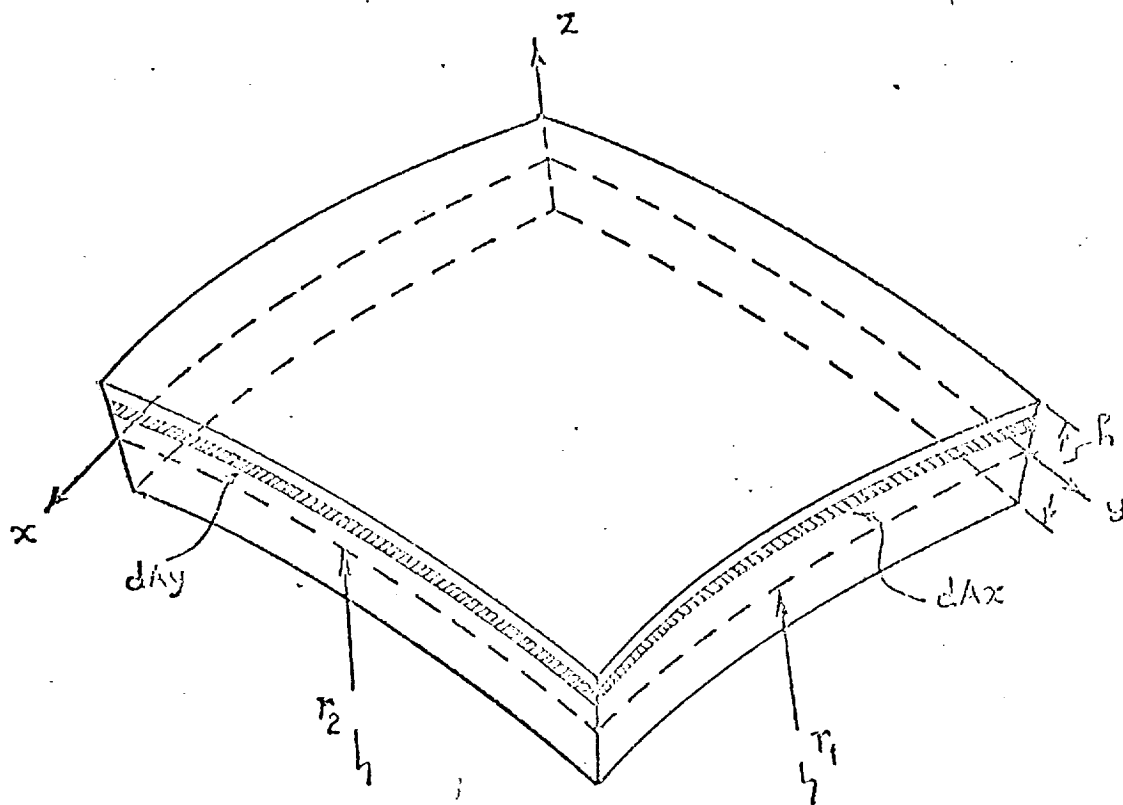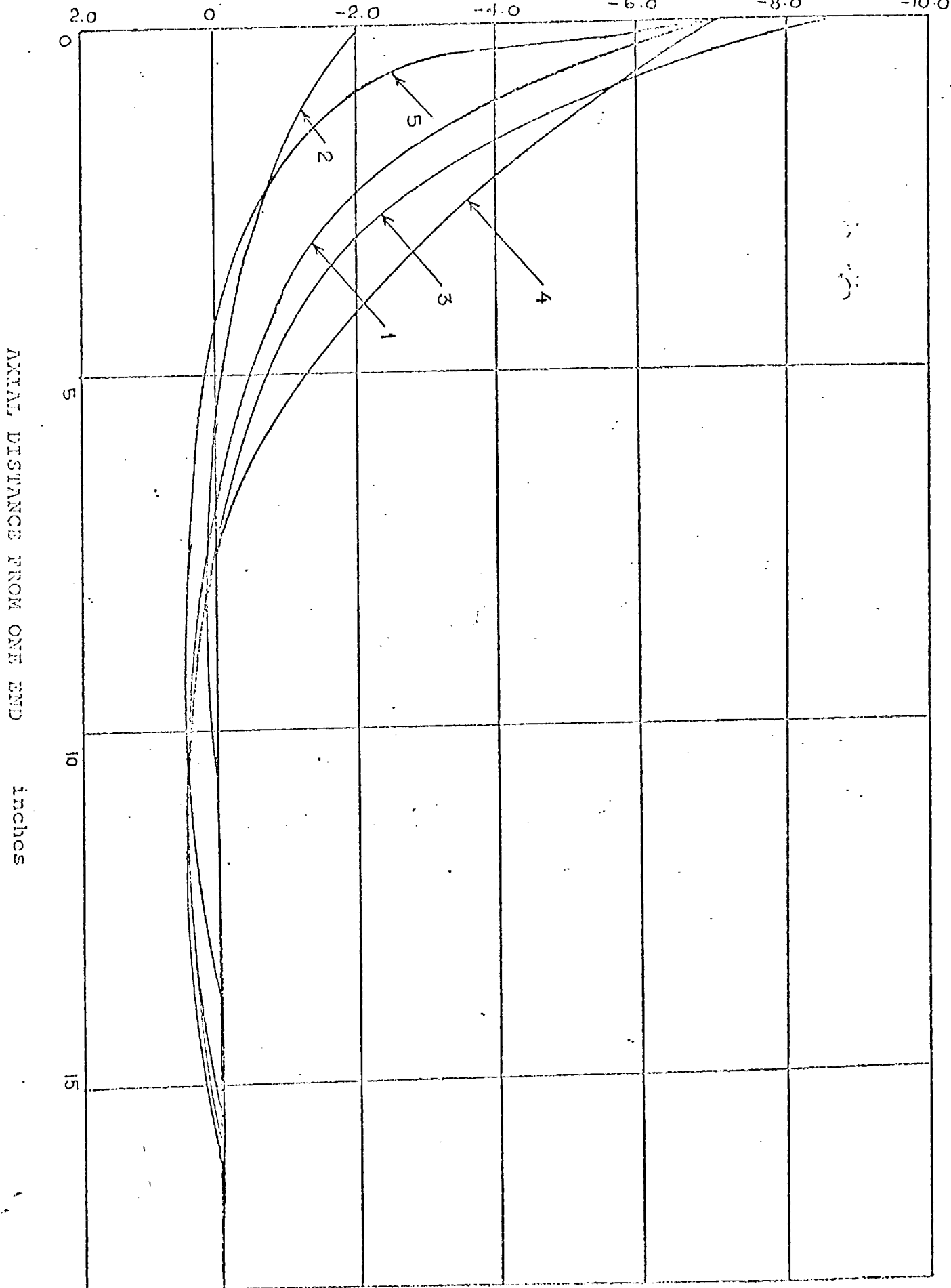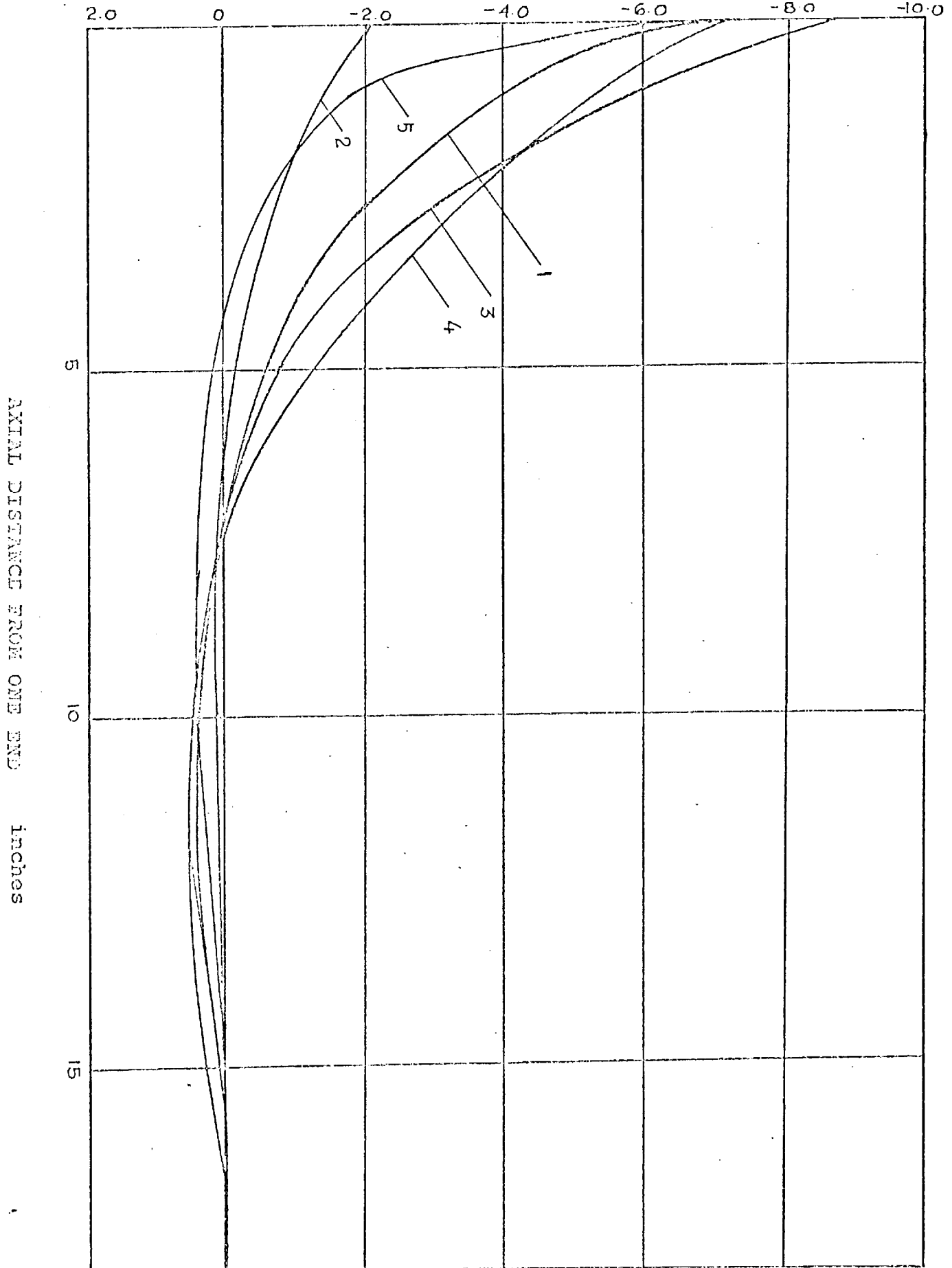
CRITICAL TEMPERATURE T_{lcr}

LENGTH / RADIUS

FIGURE 8.3

CRITICAL TEMPERATURE T_{1cr}

CRITICAL TEMPERATURE T_{lcr}

(MIDDLE SURFACE TEMP.−OUTER SURFACE TEMP.)/MIDDLE SURFACE TEMP.

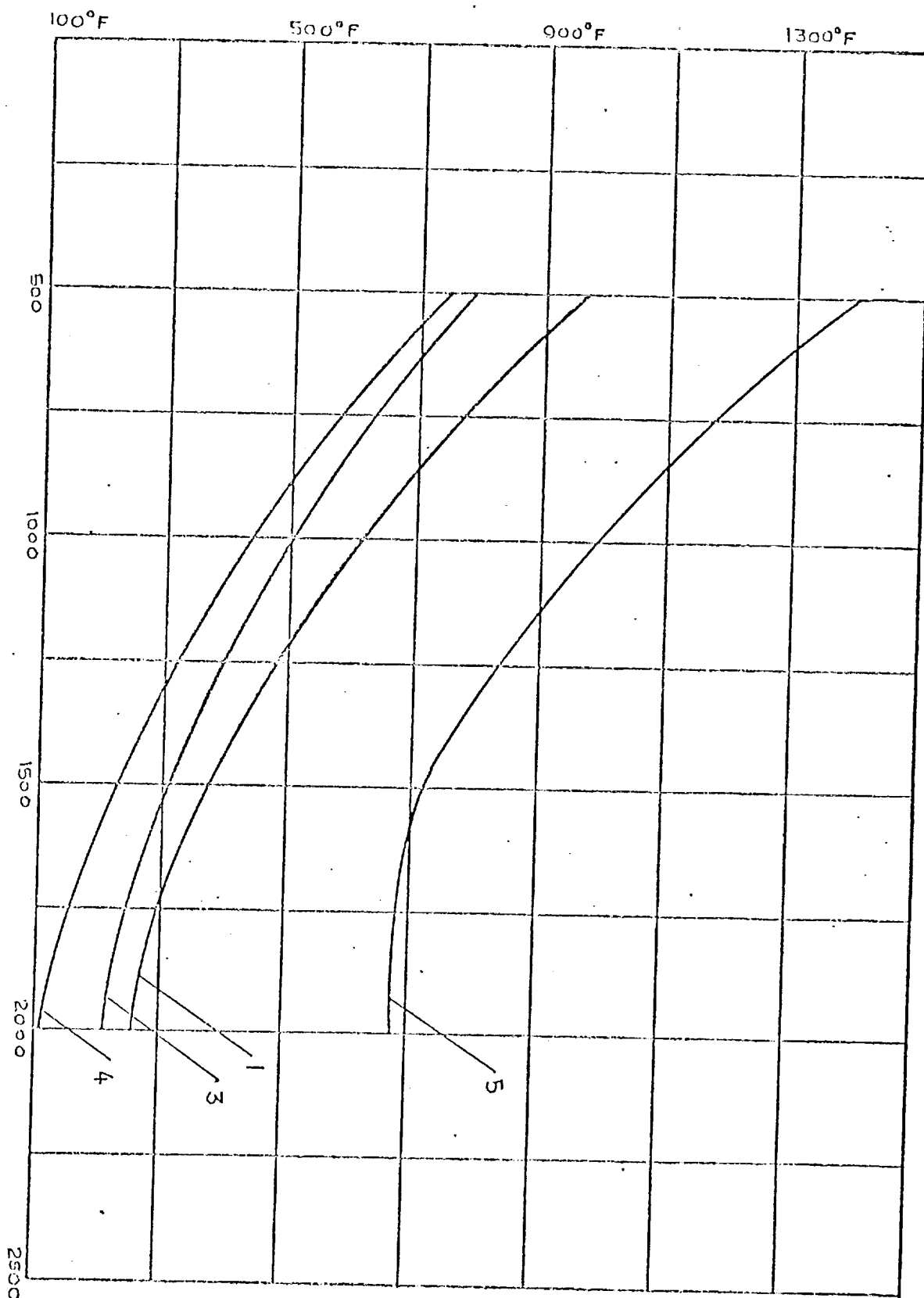50°F   150°F   0°F   350°F

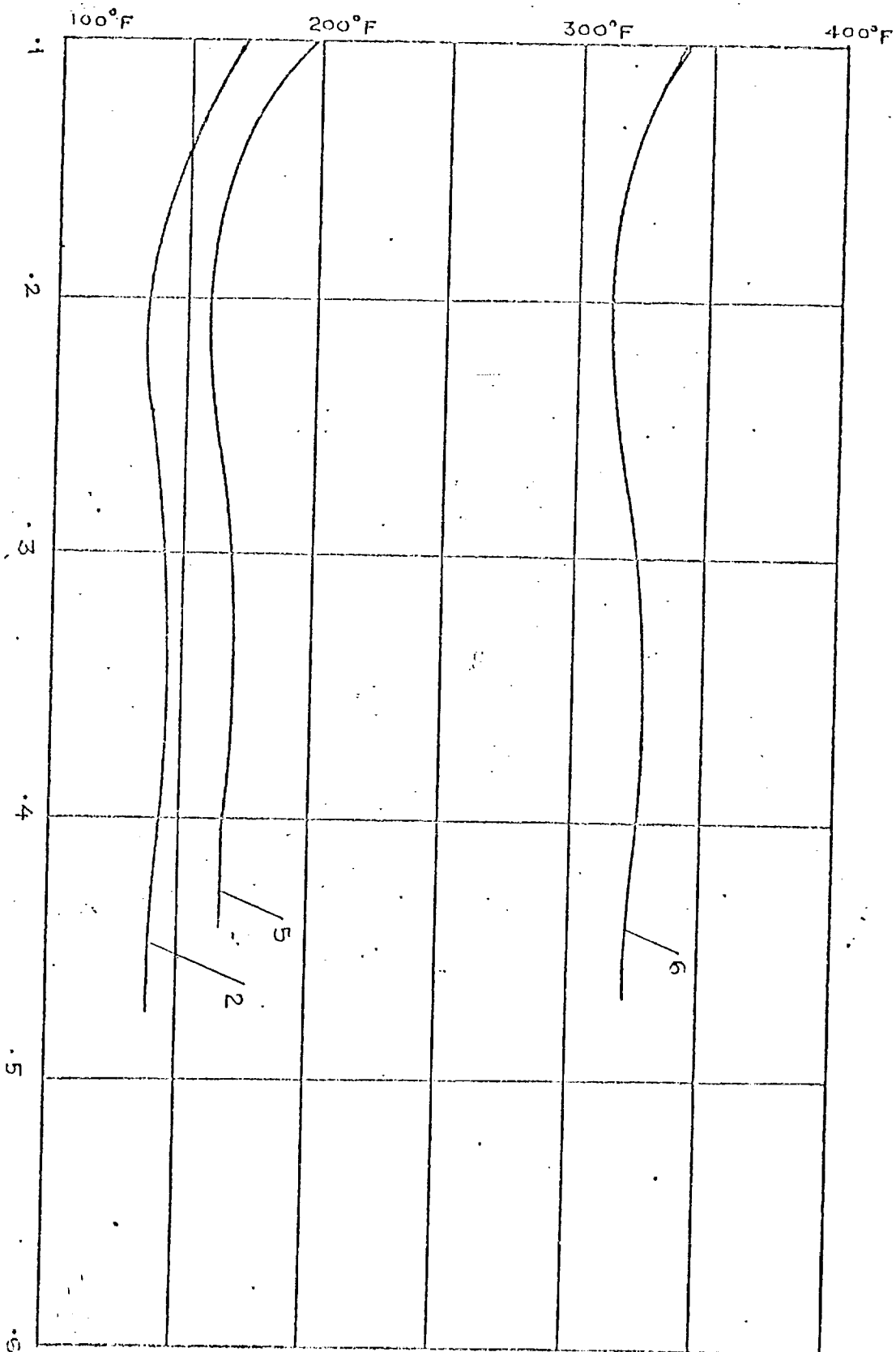0   0.05   0.10   0.15   0.20   0.25

4

2

1
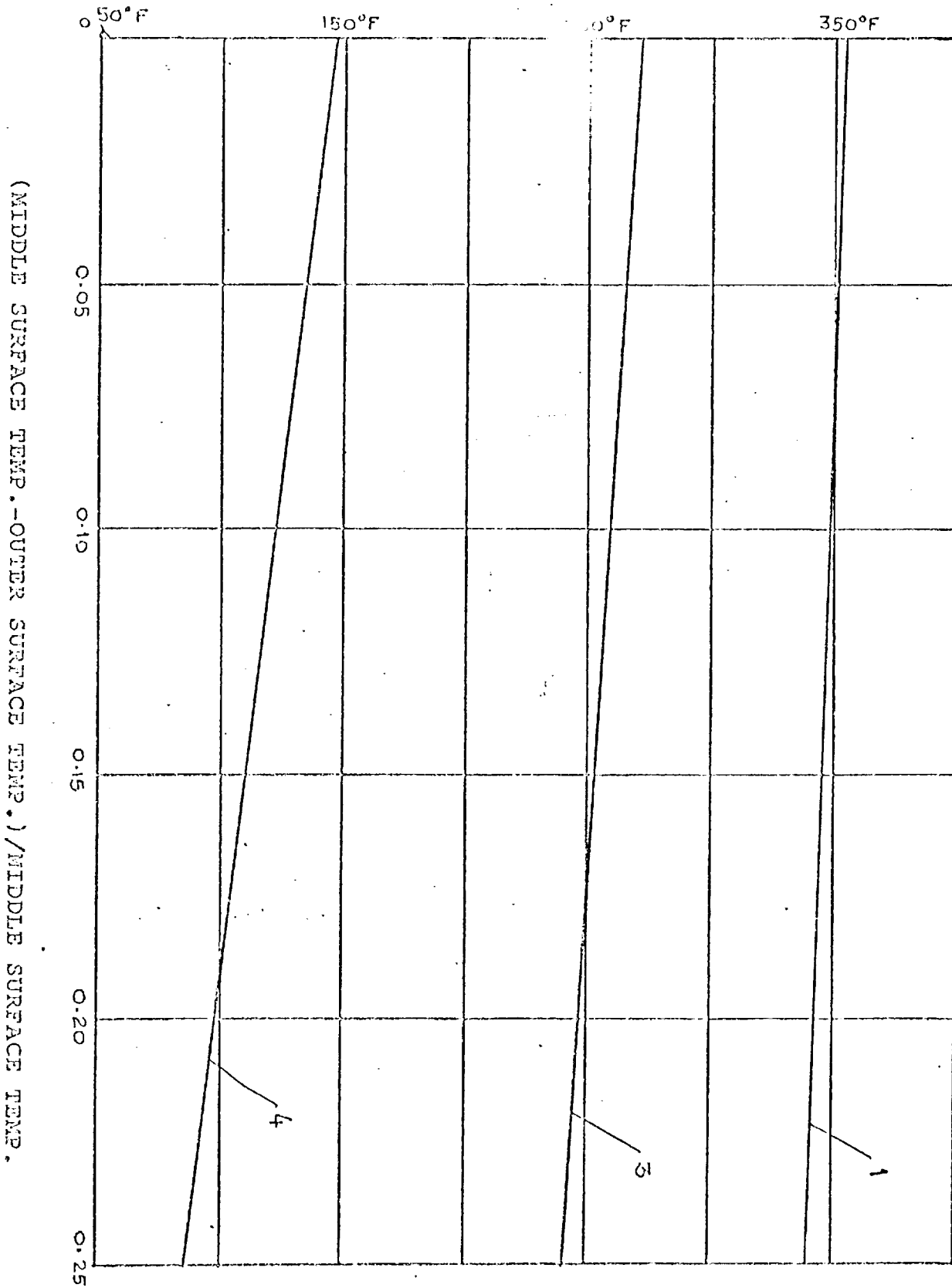
CRITICAL TEMPERATURE T_{cr}
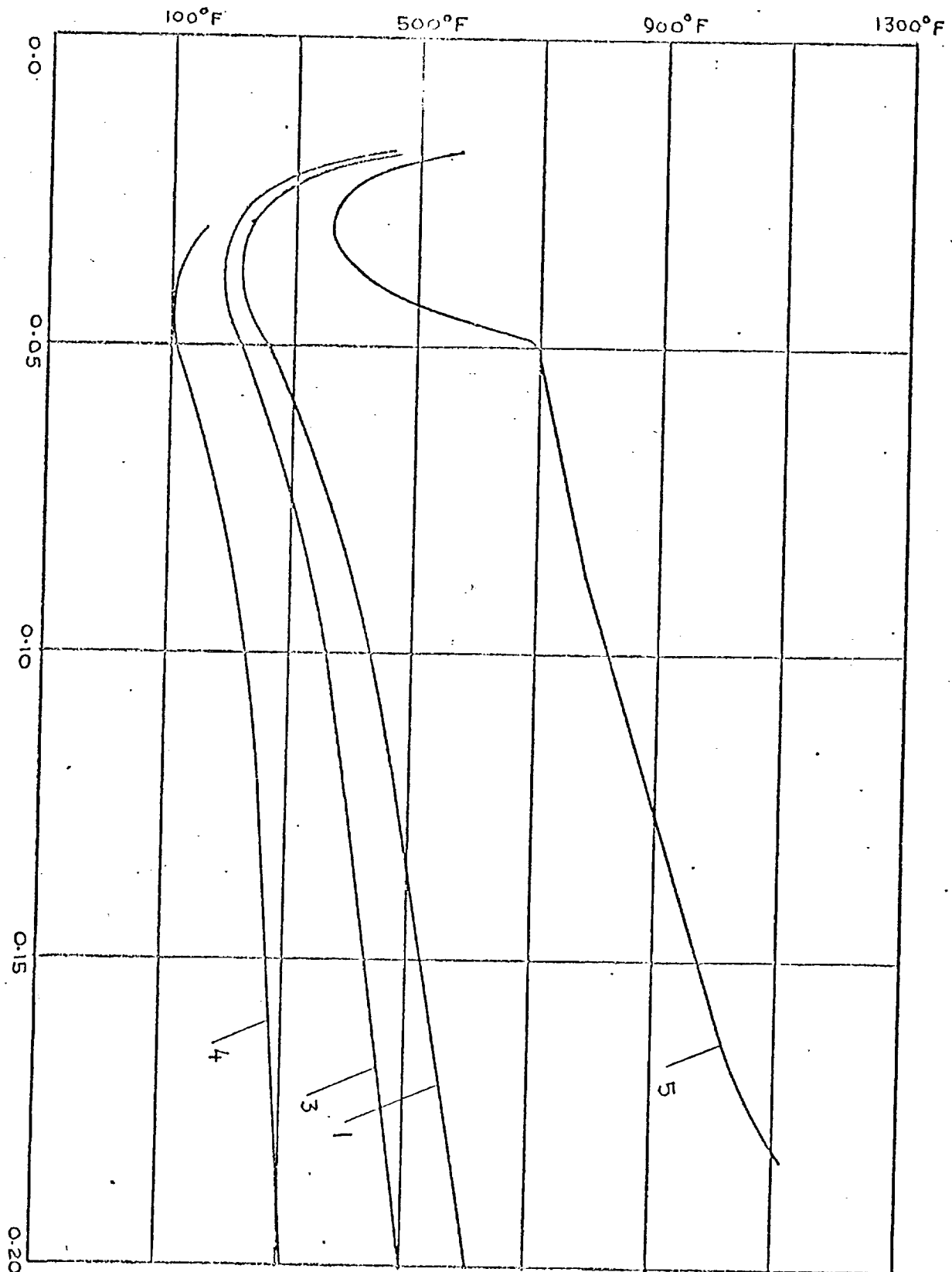
CRITICAL TEMPERATURE T_{cr}

CRITICAL TEMPERATURE T_{cr}

LENGTH / RADIUS

FIGURE 6.6

CRITICAL TEMPERATURE T_{cr}